Improper Integrals Practice

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Starred problems are challenges.

Determine if the following integrals converge, and if so, evaluate them:

1. $\displaystyle\int_0^4 \frac{1}{\sqrt{4-x}}\,dx$

2. $\displaystyle\int_0^{\pi/2} \tan x\,dx$

3. $\displaystyle\int_0^{\infty} \frac{x}{(1+x^2)^2}\,dx$

4. $\displaystyle\int_{-\infty}^{\infty} xe^{-x^2}\,dx$

5. $\displaystyle\int_{-1}^{1} \frac{1}{x^{1/3}}\,dx$

6. $\displaystyle\int_4^{\infty} \frac{1}{(x-2)(x-3)}\,dx$

Determine the convergence or divergence of the following integrals:

7. $\displaystyle\int_1^{\infty} e^{-(x+x^{-1})}\,dx$

8. $\displaystyle\int_{-\infty}^{\infty} e^{-x^2}\,dx$

9. $\displaystyle\int_0^{1} \frac{1}{x^4+\sqrt{x}}\,dx$

10. $\displaystyle\int_0^{1/2} \frac{1}{x^2\ln(x)}\,dx$

11. $\displaystyle\int_0^{\infty} \frac{1}{x^{3/2}(x+1)}\,dx$

12. $\displaystyle\int_0^{1} \frac{e^x}{x^2}\,dx$

13. $\displaystyle\int_1^{\infty} \frac{1}{x^4+e^x}\,dx$

14. $\displaystyle\int_0^{1} \frac{1}{xe^x+x^2}\,dx$

15.* $\displaystyle\int_1^{\infty} \frac{\sin(x)}{x}\,dx$ *(Hint: integrate by parts.)*

16.* $\displaystyle\int_0^{1} \frac{\arctan(x)}{e^x-1}\,dx$ *(Hint: use first order Taylor polynomials centered at* 0 *of the numerator and denominator to estimate the integrand.)*

Solutions

1. The integrand has a singularity at $x = 4$, so it is indeed improper. Write $\int_0^4 \frac{1}{\sqrt{4-x}}\,dx = \lim_{R\to 4^-} \int_0^R \frac{1}{\sqrt{4-x}}\,dx$. This integral can be done with the substitution $u = 4-x$ and $du = -\,dx$. As an indefinite integral, $\int \frac{1}{\sqrt{4-x}}\,dx = -\int u^{-1/2}\,du = -2u^{1/2} + C = -2(4-x)^{1/2} + C$. The integral becomes $\lim_{R\to 4^-} -2(4-x)^{1/2}\Big|_0^R = \lim_{R\to 4^-} -2(4-R)^{1/2} + 4 = \boxed{4}$.

2. The integrand has a singularity at $x = \pi/2$, so it is improper. Write this as $\lim_{R\to \pi/2^-} \int_0^R \tan x\,dx$. We know that $\int \tan x\,dx = -\ln|\cos x| + C$, so this becomes $\lim_{R\to\pi/2^-} -\ln|\cos x|\Big|_0^R = \lim_{R\to\pi/2^-} -\ln|\cos R| = \infty$, so that the integral $\boxed{\text{diverges}}$.

3. The integral has an infinite bound, so it's improper. Write $\int_0^\infty \frac{x}{(1+x^2)^2}\,dx$ as $\lim_{R\to\infty} \int_0^R \frac{x}{(1+x^2)^2}\,dx$. The indefinite integral $\int \frac{x}{(1+x^2)^2}\,dx$ can be done with the substitution $u = 1+x^2$ and $du = 2x\,dx$. This becomes $\frac{1}{2}\int \frac{1}{u^2}\,du = -\frac{1}{2u} + C = -\frac{1}{2(1+x^2)} + C$. We then get $\lim_{R\to\infty} -\frac{1}{2(1+x^2)}\Big|_0^R = \lim_{R\to\infty} -\frac{1}{2(1+R^2)} + \frac{1}{2} = \boxed{\frac{1}{2}}$.

4. This is a doubly infinite integral, so we need to split it up as $\int_{-\infty}^0 xe^{-x^2}\,dx + \int_0^\infty xe^{-x^2}\,dx$. Both of these are improper, so write these as $\lim_{R\to-\infty}\int_R^0 xe^{-x^2}\,dx + \lim_{B\to\infty}\int_0^B xe^{-x^2}\,dx$. The indefinite integral $\int xe^{-x^2}\,dx = -\frac{e^{-x^2}}{2} + C$ after the substitution $u = x^2$ and $du = 2x\,dx$. The first improper integral becomes $\lim_{R\to-\infty} -\frac{e^{-x^2}}{2}\Big|_R^0 = -\frac{1}{2} + \frac{e^{-R^2}}{2} = -\frac{1}{2}$. The other integral becomes $\lim_{B\to\infty} -\frac{e^{-x^2}}{2}\Big|_0^B = -\frac{e^{-B^2}}{2} + \frac{1}{2} = \frac{1}{2}$. Adding these together gives $\int_{-\infty}^\infty xe^{-x^2}\,dx = \boxed{0}$.

5. The integrand has a singularity at $x = 0$, so we need to split it up into $\int_{-1}^0 \frac{1}{x^{1/3}}\,dx + \int_0^1 \frac{1}{x^{1/3}}\,dx$. We have $\int \frac{1}{x^{1/3}}\,dx = \frac{3}{2}x^{2/3} + C$ by the power rule. This gives $\int_{-1}^0 \frac{1}{x^{1/3}}\,dx = \lim_{R\to 0^-}\int_{-1}^R \frac{1}{x^{1/3}}\,dx = \lim_{R\to 0^-} \frac{3}{2}x^{2/3}\Big|_{-1}^R = \lim_{R\to 0^-} \frac{3}{2}R^{2/3} - \frac{3}{2} = -\frac{3}{2}$. The other integral similarly can be computed as $\frac{3}{2}$, so adding them together gives $\boxed{0}$.

6. The integral has no singularity, but it has an infinite limit so it is improper. Write $\int_4^\infty \frac{1}{(x-2)(x-3)}\,dx = \lim_{R\to\infty}\int_4^R \frac{1}{(x-2)(x-3)}\,dx$. To calculate the indefinite integral $\int \frac{1}{(x-2)(x-3)}\,dx$, we use partial fractions. Write $\frac{1}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3}$. Clearing denominators gives $1 = A(x-3) + B(x-2)$. Plugging in $x = 2$ and $x = 3$ respectively gives $A = -1$ and $B = 1$, so $\frac{1}{(x-2)(x-3)} = \frac{-1}{x-2} + \frac{1}{x-3}$. This gives $\int \frac{1}{(x-2)(x-3)}\,dx = \int \frac{-1}{x-2} + \frac{1}{x-3}\,dx = -\ln|x-2| + \ln|x-3| + C$. The improper integral then becomes $\lim_{R\to\infty} -\ln|x-2| + \ln|x-3|\Big|_4^R =$

$\lim_{R\to\infty} \ln\left|\frac{x-3}{x-2}\right|\Big|_4^R = \lim_{R\to\infty} \ln\left|\frac{R-3}{R-2}\right| + \ln 2$. As $\lim_{R\to\infty} \frac{R-3}{R-2} = 1$, this gives $\lim_{R\to\infty} \ln\left|\frac{R-3}{R-2}\right| = 0$, so $\int_4^\infty \frac{1}{(x-2)(x-3)}\,dx = \boxed{\ln 2}$.

7. Note that $x + x^{-1} = x + \frac{1}{x} > x$, and because the exponential function is increasing, this says $e^{(x+x^{-1})} > e^x$. Then $e^{-(x+x^{-1})} < e^{-x}$, so $\int_1^\infty e^{-(x+x^{-1})}\,dx < \int_1^\infty e^{-x}\,dx$. One can check that $\int_1^\infty e^{-x}\,dx = \frac{1}{e}$ so by the comparison test, $\int_1^\infty e^{-(x+x^{-1})}\,dx$ $\boxed{\text{converges}}$.

8. Since e^{-x^2} is an even function, if $\int_0^\infty e^{-x^2}\,dx$ exists, then so does $\int_{-\infty}^0 e^{-x^2}\,dx$ and they have the same value, so we will just prove that $\int_0^\infty e^{-x^2}\,dx$ exists. Since $e^x \le e^{x^2}$ for all $x \ge 1$, this says $e^{-x^2} \le e^{-x}$ for $x \ge 1$. Write $\int_0^\infty e^{-x^2}\,dx = \int_0^1 e^{-x^2}\,dx + \int_1^\infty e^{-x^2}\,dx$. The above says $\int_1^\infty e^{-x^2}\,dx \le \int_1^\infty e^{-x}\,dx$, and the latter integral converges to $\frac{1}{e}$. The integral $\int_0^1 e^{-x^2}\,dx$ is finite because e^{-x^2} is continuous, and therefore bounded on this interval. By the comparison test, $\int_0^\infty e^{-x^2}\,dx$ therefore converges, and thus $\int_{-\infty}^\infty e^{-x^2}\,dx$ $\boxed{\text{converges}}$.

9. There is a singularity at $x = 0$ in the integrand, so we need to study the behavior of $\frac{1}{x^4+\sqrt{x}}$ near there. Around $x = 0$, the term that makes the most contribution to $x^4 + \sqrt{x}$ is $\sqrt{x}$, so we expect that $\int_0^1 \frac{1}{x^4+\sqrt{x}}\,dx$ and $\int_0^1 \frac{1}{\sqrt{x}}\,dx$ have the same behavior. One can check that $\int_0^1 \frac{1}{\sqrt{x}}\,dx = 2$, so let's try and get this as an upper bound. We have $x^4 + \sqrt{x} > \sqrt{x}$, so that $\frac{1}{x^4+\sqrt{x}} < \frac{1}{\sqrt{x}}$. This says $\int_0^1 \frac{1}{x^4+\sqrt{x}}\,dx < \int_0^1 \frac{1}{\sqrt{x}}\,dx$, so by the comparison test, $\int_0^1 \frac{1}{x^4+\sqrt{x}}\,dx$ $\boxed{\text{converges}}$.

10. The integrand is not defined at $x = 0$. In fact, there's a question of whether or not the integral is even infinite at $x = 0$! Determining this is a basic L'Hopital's rule computation: $\lim_{x\to 0} \frac{1}{x^2\ln(x)} = -\infty$, so we have an infinite discontinuity at $x = 0$ and have to determine how the integrand blows up. As $x \to 0$, the dominant term in the denominator has to be $\ln(x)$, because it's what causes the blowup in the first place. We'd then ideally like to compare with $\int_0^{1/2} \frac{1}{\ln(x)}\,dx$, but this isn't much of an improvement, because we can't integrate that function directly. The strategy then, is to get rid of the logarithm to compare with something easier: since logarithms eventually grow slower than any power of x, note that $\ln(x) < x$ for all $x > 0$, which means that $x^2\ln(x) < x^3$ for all $x > 0$. Inverting says that $\frac{1}{x^2\ln(x)} > \frac{1}{x^3}$ for $x > 0$, and since $\int_0^{1/2} \frac{1}{x^3}\,dx$ diverges, the comparison test then says that $\int_0^{1/2} \frac{1}{x^2\ln(x)}\,dx$ also $\boxed{\text{diverges}}$.

11. The integrand has both a singularity at $x = 0$ and an infinite limit. We will split it up into a sum to analyze it better. Write $\int_0^\infty \frac{1}{x^{3/2}(x+1)}\,dx = \int_0^1 \frac{1}{x^{3/2}(x+1)}\,dx + \int_1^\infty \frac{1}{x^{3/2}(x+1)}\,dx$. In the first integral, near $x = 0$, $x^{3/2}(x+1) = x^{5/2} + x^{3/2}$ looks like $x^{3/2}$, so we expect that $\int_0^1 \frac{1}{x^{3/2}(x+1)}\,dx$ and $\int_0^1 \frac{1}{x^{3/2}}\,dx$ have the same behavior. We know $\int_0^1 \frac{1}{x^{3/2}}\,dx$ diverges, so let's try and use this as a lower bound. We have $x^{3/2}(x+1) < 2x^{3/2}$, so indeed this gives

$\frac{1}{x^{3/2}(x+1)} > \frac{2}{x^{3/2}}$. Integrating both sides then shows that $\int_0^1 \frac{1}{x^{3/2}(x+1)}\,dx$ diverges. For the second piece $\int_1^\infty \frac{1}{x^{3/2}(x+1)}\,dx$, as x goes to infinity, $x^{3/2}(x+1) = x^{5/2} + x^{3/2}$ behaves like $x^{5/2}$, so we expect that $\int_1^\infty \frac{1}{x^{3/2}(x+1)}\,dx$ and $\int_1^\infty \frac{1}{x^{5/2}}\,dx$ have the same behavior. The latter integral converges, so we try to use it as an upper bound. We see $x^{3/2}(x+1) > x^{5/2}$, so that $\frac{1}{x^{3/2}(x+1)} < \frac{1}{x^{5/2}}$. This gives $\int_1^\infty \frac{1}{x^{3/2}(x+1)}\,dx < \int_1^\infty \frac{1}{x^{5/2}}\,dx$, so that by the comparison test, $\int_1^\infty \frac{1}{x^{3/2}(x+1)}\,dx$ converges. Since the first piece diverges and the second piece converges, we find that $\int_0^\infty \frac{1}{x^{3/2}(x+1)}\,dx$ $\boxed{\text{diverges}}$.

12. There is a singularity at $x = 0$, so we need to analyze the integrand there. On the interval $[0, 1]$ the function e^x is bounded and therefore makes little contribution, so we expect $\int_0^1 \frac{e^x}{x^2}\,dx$ and $\int_0^1 \frac{1}{x^2}\,dx$ to have the same behavior. As $1 \le e^x$ for $0 \le x \le 1$, this says $\frac{e^x}{x^2} \ge \frac{1}{x^2}$. Integrating both sides gives $\int_0^1 \frac{1}{x^2}\,dx \le \int_0^1 \frac{e^x}{x^2}\,dx$. Since $\int_0^1 \frac{1}{x^2}\,dx$ diverges, by the comparison test we must have that $\int_0^1 \frac{e^x}{x^2}\,dx$ $\boxed{\text{diverges}}$.

13. As x goes to infinity, because e^x grows faster than x^4, we see e^x has the biggest contribution in $x^4 + e^x$. So we expect that $\int_1^\infty \frac{1}{x^4+e^x}\,dx$ and $\int_1^\infty \frac{1}{e^x}\,dx$ have the same behavior. We have $x^4 + e^x > e^x$, so that $\frac{1}{x^4+e^x} < \frac{1}{e^x}$. Integrating gives $\int_1^\infty \frac{1}{x^4+e^x}\,dx < \int_1^\infty \frac{1}{e^x}\,dx$, and one can check that $\int_1^\infty \frac{1}{e^x}\,dx$ does in fact converge. By the comparison test, this says $\int_1^\infty \frac{1}{x^4+e^x}\,dx$ $\boxed{\text{converges}}$.

14. There is a singularity at $x = 0$, in the integrand, so we need to analyze the behavior of $\frac{1}{xe^x + x^2}$ near there. Note that e^x is bounded on $[0, 1]$ so it does not contribute much to the term $xe^x + x^2$. In fact, $e^x \approx 1$ near 0, so $xe^x + x^2 \approx x + x^2$. How does $x + x^2$ behave near $x = 0$? The term with the largest contribution is x, so $x + x^2 \approx x$ and we then expect that $\int_0^1 \frac{1}{xe^x+x^2}\,dx$ and $\int_0^1 \frac{1}{x}\,dx$ have the same behavior. Since $\int_0^1 \frac{1}{x}\,dx$ diverges, we try to use this as a lower bound. For $0 \le x \le 1$, we have $e^x \le e$, so that $xe^x + x^2 \le xe + x^2$. As $x^2 \le x$ on this interval as well, $xe^x + x^2 \le xe + x = x(e + 1)$. This says $xe^x + x^2 \le x(e + 1)$, so that $\frac{1}{xe^x+x^2} \ge \frac{1}{x(e+1)}$. Integrating this inequality says $\frac{1}{e+1}\int_0^1 \frac{1}{x}\,dx \le \int_0^1 \frac{1}{xe^x+x^2}\,dx$, so by the comparison test, $\int_0^1 \frac{1}{xe^x+x^2}\,dx$ $\boxed{\text{diverges}}$.

15. Write $\int_1^\infty \frac{\sin(x)}{x}\,dx = \lim_{R\to\infty} \int_1^R \frac{\sin(x)}{x}\,dx$. Using integration by parts with $u = \frac{1}{x}$ and $dv = \sin(x)$, we find $\int_1^R \frac{\sin(x)}{x}\,dx = -\frac{\cos(x)}{x}\Big|_1^R - \int_1^R \frac{\cos(x)}{x^2}\,dx = \cos(1) - \frac{\cos(R)}{R} - \int_1^R \frac{\cos(x)}{x^2}\,dx$. As $R \to \infty$, $\frac{\cos(R)}{R} \to 0$ by the squeeze theorem, and $\lim_{R\to\infty} \int_1^R \frac{\cos(x)}{x^2}\,dx = \int_1^\infty \frac{\cos(x)}{x^2}\,dx \le \int_1^\infty \frac{|\cos(x)|}{x^2}\,dx \le \int_1^\infty \frac{1}{x^2}\,dx < \infty$. This says $\int_1^\infty \frac{\sin(x)}{x}\,dx = \lim_{R\to\infty} \int_1^R \frac{\sin(x)}{x}\,dx = \cos(1) - \int_1^\infty \frac{\cos(x)}{x^2}\,dx$ is finite, so the integral $\boxed{\text{converges}}$.

16. Near $x = 0$, $e^x \approx 1 + x$, so $e^x - 1 \approx x$, and similarly we have $\arctan(x) \approx x$, so near $x = 0$, $\dfrac{\arctan(x)}{e^x - 1} \approx 1$, and therefore we expect the integral converges. To apply the comparison test, we will show that $\dfrac{\arctan(x)}{e^x - 1} \leq 1$ for all x. To see this, the trick is to instead look at derivatives: $\dfrac{1}{x^2+1} \leq 1 \leq e^x$ for all x, so integrating yields $\arctan(x) = \displaystyle\int_0^x \frac{1}{t^2+1}\,dt \leq \int_0^x e^t\,dt = e^x - 1$, i.e. $\dfrac{\arctan(x)}{e^x - 1} \leq 1$. This then gives $\displaystyle\int_0^1 \frac{\arctan(x)}{e^x - 1}\,dx \leq \int_0^1 1\,dx = 1$, so by the comparison test, $\displaystyle\int_0^1 \frac{\arctan(x)}{e^x - 1}\,dx$ $\boxed{\text{converges}}$.