INFINITE SERIES

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These notes are a treatment of the standard convergence tests for infinite series as covered in a second semester calculus course like Math 31B at UCLA. There may be many typos – please let me know if any are found!

1. Introduction

Definition 1.1. A **sequence** is a function $f : \mathbb{N} \to \mathbb{R}$, where $\mathbb{N}$ is the set of non-negative integers.

We usually write a_n to denote the value $f(n)$ of the function f, because we like to think of sequences as different types of objects than functions. Often times, it's useful to think about a sequence as it's set of values $\{a_n\}$, and we typically write $\{a_n\}$ to refer to the sequence instead of f.

Example 1.2. We can think of the sequence $a_n = \frac{1}{n^2}$ as either being some object explicitly defined by the above formula, or as the list of values $\{1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \ldots\}$.

Example 1.3. Define a sequence by $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$. This is an example of a **recursive sequence**, a sequence where the value at some given n depends on the previous terms. Explicitly, the first few terms of this sequence are $1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$. This sequence is called the **Fibonacci sequence**.

Since sequences are functions, all the usual operations you're used to doing with functions make sense for sequences: addition/subtraction, multiplication/division, taking limits, etc. Ultimately, our goal is to understand how calculus works in the discrete world, with sequences taking the role of functions. Below are the analogies between calculus in $\mathbb{R}$ and discrete calculus that should be kept in mind to strengthen conceptual understanding.

Calculus in $\mathbb{R}$	Discrete calculus
Functions $f : \mathbb{R} \to \mathbb{R}$	Sequences $\{a_n\}$
Derivative: $\frac{d}{dx} f(x)$	Forward difference: $\Delta a_n = a_{n+1} - a_n$
Anti-derivative: $\int f(x)\,dx$	Partial sum: $\sum_{n=1}^{N} a_n$
Definite integral: $\int_a^b f(x)\,dx$	Sum: $\sum_{n=a}^{b} a_n$
Improper integral: $\int_1^\infty f(x)\,dx$	Infinite series: $\sum_{n=1}^{\infty} a_n$

2. Basic definitions

Definition 2.1. Given a sequence $\{a_n\}$, define a new sequence $\{S_N\}$ by $S_N = a_1 + a_2 + \ldots + a_N = \sum_{n=1}^{N} a_n$. The sequence $\{S_N\}$ is called the sequence of **partial sums** of $\{a_n\}$. An **infinite series** is an expression of the form $\sum_{n=1}^{\infty} a_n$, i.e. addition of infinitely many terms of some sequence (for convenience the starting index is 1, but it does not matter).

Date:

As is usual in calculus, to try and understand something "infinite", we have to take limits of "finite" things that we understand. Analogously to how we define $\int_1^\infty f(x)\,dx$ through limits of definite integral $\lim_{R\to\infty}\int_1^R f(x)\,dx$, we will define an infinite series by taking a limit of its partial sums.

Definition 2.2. We say $\sum_{n=1}^\infty a_n$ **converges** if $\lim_{N\to\infty} S_N$ is finite, and if $\lim_{N\to\infty} S_N = L$, we say $\sum_{n=1}^\infty a_n = L$. The infinite series $\sum_{n=1}^\infty a_n$ **diverges** if $\lim_{N\to\infty} S_N$ does not exist.

Series can be confusing at first because of the different types of objects involved. A series is a formal infinite expression of the form $a_1+a_2+\ldots$. A series can be assigned a value, which is obtained by taking a limit of a sequence (the sequence of partial sums). In particular, don't mix up sequences with series: series have a value, sequences are functions.

Example 2.3. Set $a_n = \frac{1}{n}$, so that $S_N = \sum_{n=1}^N \frac{1}{n}$. The first few terms of the sequence S_N are given by $1, \frac{3}{2}, \frac{11}{6}, \frac{25}{12}, \frac{137}{60}, \ldots$. The series $\sum_{n=1}^\infty \frac{1}{n}$ is called the **harmonic series**. It turns out that $\sum_{n=1}^\infty \frac{1}{n}$ diverges (which is not at all obvious).

Example 2.4. Set $a_n = n$. Then the sequence of partial sums $\{S_N\}$ has terms given by $S_N = \sum_{n=1}^N n = \frac{N(N+1)}{2}$. Since $\lim_{N\to\infty} S_N = \infty$, the series $\sum_{n=1}^\infty n$ diverges.

Example 2.5. Let $\{a_n\}$ be a sequence such that the N-th partial sum is given by $S_N = 3 - \frac{1}{N^2}$. Since $\lim_{N\to\infty} S_N = 3$, this says $\sum_{n=1}^\infty a_n$ converges, and we have $\sum_{n=1}^\infty a_n = 3$. Notice we know nothing about the actual terms in the sequence $\{a_n\}$ – the definition of convergence or divergence of an infinite series depends only on the partial sums.

Example 2.6. With $S_N = 3 - \frac{1}{N^2}$ as above, we can recover what the general term of the sequence is. Taking a forward difference, we have $\Delta S_N = S_{N+1} - S_N = \sum_{n=1}^{N+1} a_n - \sum_{n=1}^N a_n = a_{N+1}$, so $a_{N+1} = \frac{1}{N^2} - \frac{1}{(N+1)^2}$. Re-indexing, we find $a_n = \frac{1}{(n-1)^2} - \frac{1}{n^2}$ for $n \geq 2$, and $S_1 = a_1 = 2$. This process is analogous to how a function can be recovered from knowledge of it's anti-derivative by differentiating.

3. Geometric and Telescoping series

Definition 3.1. A **geometric series** is an infinite series of the form $\sum_{n=M}^\infty ar^n$ for some non-zero real numbers a and r, and some starting index M.

Geometric series are "simple" series in the sense that we can classify their behavior completely:

Theorem 3.2 (**Classification of geometric series**)**.** *If $|r| < 1$, then $\sum_{n=M}^\infty ar^n$ converges, and $\sum_{n=M}^\infty ar^n = \frac{ar^M}{1-r}$. Otherwise if $|r| \geq 1$, then $\sum_{n=M}^\infty ar^n$ diverges.*

Example 3.3. The series $\sum_{n=1}^\infty 5(\frac{1}{2})^n$ is a geometric series with $a = 5$, $r = \frac{1}{2}$, and $M = 1$. We see $\sum_{n=1}^\infty 5(\frac{1}{2})^n = \frac{5/2}{1-1/2} = 5$.

Example 3.4. Consider the infinite series $\sum_{n=0}^\infty \frac{3\cdot 2^{2n-2} + (-1)^n 5^{n+1}}{6^n}$. Splitting this up, we can write this as $\sum_{n=0}^\infty \frac{3\cdot 2^{2n-2}}{6^n} + \sum_{n=0}^\infty (-1)^n \frac{5^{n+1}}{6^n}$. Using exponent rules to write each sum as a geometric series, we find $\sum_{n=0}^\infty \frac{3\cdot 2^{2n-2}+5^{n+1}}{6^n} = \sum_{n=0}^\infty \frac{3}{4}(\frac{4}{6})^n + \sum_{n=0}^\infty 5(-\frac{5}{6})^n = \frac{3/4}{1-2/3} + \frac{5}{1+5/6} = \frac{219}{44}$ using the above formula.

Definition 3.5. A **telescoping series** is an infinite series of the form $\sum_{n=1}^{\infty}(a_{n+1} - a_n)$ for some sequence $\{a_n\}$.

The name telescoping comes from writing down the summation of the terms in the series – they cancel out and "collapse" like a telescope. The N-th partial sum of a telescoping series is $S_N = \sum_{n=1}^{N}(a_{n+1} - a_n) = (a_2 - a_1) + (a_3 - a_2) + (a_4 - a_3) + \ldots + (a_N - a_{N-1}) + (a_{N+1} - a_N) = a_{N+1} - a_1$, so taking a limit gives the following:

Theorem 3.6 (**Discrete FTC**). *Suppose that* $\lim_{n\to\infty} a_n = L$. *Then* $\sum_{n=1}^{\infty}(a_{n+1} - a_n) = L - a_1$.

Writing the above statement using the forward difference operator, the theorem says $\sum_{n=1}^{\infty} \Delta a_n = L - a_1$ where $L = \lim_{n\to\infty} a_n$. The analogue is the statement that $\int_1^{\infty} f'(x)\,dx = L - f(1)$ where $L = \lim_{x\to\infty} f(x)$, which is just the fundamental theorem of calculus (applied to improper integrals).

Example 3.7. The series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is a telescoping series. To see this, using partial fractions we can write $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$, and we then see $\sum_{n=1}^{\infty}(\frac{1}{n} - \frac{1}{n+1})$ is a telescoping series with $a_n = -\frac{1}{n}$, and $\lim_{n\to\infty} a_n = 0$, so that $\sum_{n=1}^{\infty}(\frac{1}{n} - \frac{1}{n+1}) = 1$.

Example 3.8. Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(\frac{(n+1)^n}{n^{(n+1)}})}{n(n+1)}$. Using log rules, we can write this as $\sum_{n=1}^{\infty} \frac{n\ln(n+1) - (n+1)\ln(n)}{n(n+1)} = \sum_{n=1}^{\infty}\left(\frac{\ln(n+1)}{n+1} - \frac{\ln(n)}{n}\right)$. This is a telescoping series with $a_n = \frac{\ln(n)}{n}$. As $\lim_{n\to\infty} \frac{\ln(n)}{n} = 0$, we find $\sum_{n=1}^{\infty} \frac{\ln(\frac{(n+1)^n}{n^{(n+1)}})}{n(n+1)} = 0$.

Unlike with integration where we have many different techniques and rules for explicitly computing anti-derivatives, finding a sequence b_n with $\Delta b_n = a_n$ is in general, very hard. Therefore, it's generally not very obvious if a series telescopes or not! Because this process is so difficult, it's not very easy to go through the definition of an infinite series to determine if it converges or diverges. We'll have to develop more theory to help us get around this issue.

4. The comparison tests

There is a useful test for quickly checking if a series diverges:

Theorem 4.1 (**Divergence Test**). *Let* $\{a_n\}$ *be a sequence. If* $\lim_{n\to\infty} a_n \neq 0$, *then* $\sum_{n=1}^{\infty} a_n$ *diverges.*

Example 4.2. The divergence test says the series $\sum_{n=1}^{\infty} \frac{n}{n+1}$ diverges, because $\lim_{n\to\infty} \frac{n}{n+1} = 1 \neq 0$. The series $\sum_{n=1}^{\infty}(1 + \sin(n))$ also diverges, because $\lim_{n\to\infty} 1 + \sin(n)$ does not exist.

Warning: the divergence test does **not** say that if $\lim_{n\to\infty} a_n = 0$, that $\sum_{n=1}^{\infty} a_n$ converges. As we will later see, the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, but as mentioned before the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. In both series, the general term tends to 0, so if this happens we cannot conclude anything about convergence or divergence.

Our first two series test are going to be our most powerful ones.

Theorem 4.3 (**Direct comparison test**). *Let* $\sum a_n$, $\sum b_n$ *be infinite series with* $a_n, b_n \geq 0$, *and assume that* $\sum a_n \leq \sum b_n$ *eventually.*

(a) *If* $\sum a_n$ *diverges, then* $\sum b_n$ *diverges.*
(b) *If* $\sum b_n$ *converges, then* $\sum a_n$ *converges.*

Intuitively, the direct comparison test says anything smaller than a convergent series converges (i.e. anything smaller than a finite sum is finite), and anything larger than a divergent series is divergence (i.e., anything larger than an infinite sum is infinite). Notice that we only need that the inequality on series holds **eventually**. We may always rip out a finite number of terms from the sum (which doesn't change convergence) to make such an inequality explicitly true (provided $a_n \leq b_n$ eventually holds).

Theorem 4.4 (**Limit comparison test**). *Let* $\sum a_n$*,* $\sum b_n$ *be infinite series with* $a_n \geq 0$ *and* $b_n > 0$*. Set* $L = \lim_{n\to\infty} \frac{a_n}{b_n}$ *and assume that* L *exists. If* $0 < L < \infty$*, then* $\sum a_n$ *and* $\sum b_n$ *both converge or diverge together.*

Intuitively, if L is finite, this says eventually, that $a_n \approx Lb_n$, so the terms in the series roughly differ by a constant multiple, which won't change the convergence or divergence.

Each comparison test has its own set of pros and cons. In general, the direct comparison test will be a bit harder to apply, since one needs to exhibit explicit inequalities, which might be tricky to find. The limit comparison test is typically more useful, because in the process of intuitively reasoning if a series will converge or diverge, one often gets another series to compare with for free, and computing a limit is much easier than trying determine which series is larger. The direct comparison tests is more useful in a few specific cases: when the series has terms with logarithms (which grow too slowly to find a different series with similar growth speed), or with exponentials (which grow too quickly). Another situation where the direct comparison test is useful is when trigonometric functions like sine or cosine appear, as we have explicit upper/lower bounds on these functions. When we later cover Taylor series, we will see how to come up with good approximations to these types of functions that allow the limit comparison test to more easily apply.

Before moving onto examples, we need some series whose behavior is known that we can compare to. Above we classified the convergence or divergence of geometric series. Another common family of series, known as p-series, have the following behavior:

Theorem 4.5 (**Classification of p-series**). *The* p*-series* $\sum_{n=1}^{\infty} \frac{1}{n^p}$ *converges if* $p > 1$ *and diverges if* $0 < p \leq 1$*.*

It will be useful to know how quickly certain commonly encountered functions grow. We will use the notation "$a_n \ll b_n$" to mean the sequence a_n is eventually smaller than b_n, i.e. there is some N such that $a_n \leq b_n$ for all $n \geq N$. Another way of saying this is that a_n grows slower than b_n.

Theorem 4.6. *The following hold for any* $a > 0$ *and any* $b > 1$*:* $\ln(n) \ll n^a \ll b^n \ll n! \ll n^n$*.*

Remark 4.7. The above theorem is actually even stronger than what is stated. As you move up the hierarchy, not only do you have eventual inequalities, but you also have eventual limit domination, meaning that the limit of the ratio tends to 0 as $n \to \infty$. For example, $\lim_{n\to\infty} \frac{\ln(n)}{n^a} = 0$ for any $a > 0$. This strengthening of the theorem can be proved by repeated applications of L'Hopital's rule (and in fact, is really how you would prove the above version anyway.)

Example 4.8. The series $\sum_{n=1}^{\infty}\frac{1}{n2^n}$ converges. The general term $\frac{1}{n2^n}$ decays more quickly than $\frac{1}{2^n}$, and the series $\sum_{n=1}^{\infty}\frac{1}{2^n}$ converges, so we expect our series converges as well. We see that $2^n < n2^n$, so that $\frac{1}{n2^n} < \frac{1}{2^n}$ for all n. This says $\sum_{n=1}^{\infty}\frac{1}{n2^n} < \sum_{n=1}^{\infty}\frac{1}{2^n}$. The latter is a convergent geometric series, so the result follows by the direct comparison test.

Example 4.9. The series $\sum_{n=0}^{\infty}\frac{4}{4^n+n!}$ converges. The general term $\frac{1}{n!+4^n}$ decays more quickly than $\frac{1}{4^n}$, and the series $\sum_{n=0}^{\infty}\frac{1}{4^n}$ converges, so we expect our series converges as well. Since $n!+4^n > 4^n$ for all $n \geq 0$, we see that $\frac{1}{4^n+n!} < \frac{1}{4^n}$, so multiplying by 4 says $\frac{4}{4^n+n!} < \frac{4}{4^n}$ for $n \geq 0$. Since $\sum_{n=0}^{\infty}\frac{4}{4^n} = 4\sum_{n=0}^{\infty}(\frac{1}{4})^n$ is a convergent geometric series, we see that $\sum_{n=0}^{\infty}\frac{4}{4^n+n!}$ converges by a direct comparison test.

Example 4.10. The series $\sum_{n=1}^{\infty}\frac{\sqrt{n}}{n-1}$ diverges. As $n\to\infty$, $\frac{\sqrt{n}}{n-1}\approx\frac{\sqrt{n}}{n}=\frac{1}{\sqrt{n}}$. Since $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ diverges, we think our series should diverge as well. Since $n-1<n$, we get $\frac{1}{n-1}>\frac{1}{n}$ so that $\frac{\sqrt{n}}{n-1}>\frac{\sqrt{n}}{n}=\frac{1}{\sqrt{n}}$. The series $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ is a divergent p-series with $p=1/2$, so the result follows by the direct comparison test.

Example 4.11. The series $\sum_{n=1}^{\infty}\frac{1}{\sqrt[3]{n}(1+\sqrt{n})}$ diverges. As $n\to\infty$, $1+\sqrt{n}\approx\sqrt{n}$, so that $\frac{1}{\sqrt[3]{n}(1+\sqrt{n})}\approx\frac{1}{n^{5/6}}$. Since $\sum_{n=1}^{\infty}\frac{1}{n^{5/6}}$ is a divergent p-series, we expect our original series also diverges. As $1\leq\sqrt{n}$ for $n\geq 1$, we see $1+\sqrt{n}\leq\sqrt{n}+\sqrt{n}=2\sqrt{n}$, so that $\sqrt[3]{n}(1+\sqrt{n})\leq\sqrt[3]{n}(2\sqrt{n})=2n^{5/6}$. This then says $\frac{1}{\sqrt[3]{n}(1+\sqrt{n})}\geq\frac{1}{2n^{5/6}}$, and $\sum_{n=1}^{\infty}\frac{1}{2n^{5/6}}$ is a divergent p-series with $p=5/6$. The original series diverges by a direct comparison.

Example 4.12. The series $\sum_{n=1}^{\infty}\frac{1}{n-\ln(n)}$ diverges. As $n\to\infty$, the only term that matters in the denominator is n, because logarithms grow slowly. So we expect $\frac{1}{n-\ln(n)}\approx\frac{1}{n}$, which would say that $\sum_{n=1}^{\infty}\frac{1}{n-\ln(n)}$ and $\sum_{n=1}^{\infty}\frac{1}{n}$ should have the same behavior. The latter is the divergent harmonic series, so we expect our original series diverges. Set $a_n=\frac{1}{n-\ln(n)}$ and $b_n=\frac{1}{n}$. Then $\frac{a_n}{b_n}=\frac{n}{n-\ln(n)}$ and $\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{n}{n-\ln(n)}=1$ by L'Hopital's rule. Therefore by the limit comparison test, our original series diverges.

Example 4.13. The series $\sum_{n=1}^{\infty}\frac{n^3}{n^5+4n+1}$ converges. As $n\to\infty$, the fastest growing term in the denominator is n^5, so we expect $\frac{n^3}{n^5+4n+1}\approx\frac{n^3}{n^5}=\frac{1}{n^2}$. Using a limit comparison test with $a_n=\frac{n^3}{n^5+4n+1}$ and $b_n=\frac{1}{n^2}$, we see $\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{n^5}{n^5+4n+1}=1$. This says the behavior of the original series is the same as $\sum_{n=1}^{\infty}\frac{1}{n^2}$, which is a convergent p-series. The result then follows by the limit comparison test.

Example 4.14. The series $\sum_{n=1}^{\infty}\frac{e^n+n}{e^{2n}-n^2}$ converges. As $n\to\infty$, the exponential terms are the only things that matter in the numerator and denominator, because they grow the fastest. Therefore, $\frac{e^n+n}{e^{2n}-n^2}\approx\frac{e^n}{e^{2n}}=(\frac{1}{e})^n$. Therefore, we expect that $\sum_{n=1}^{\infty}\frac{e^n+n}{e^{2n}-n^2}$ and $\sum_{n=1}^{\infty}(\frac{1}{e})^n$ have the same behavior. The latter series is a convergent geometric series with $r=\frac{1}{e}$, so our original series should converge as well. Set $a_n=\frac{e^n+n}{e^{2n}-n^2}$ and $b_n=\frac{1}{e^n}$. Then $\frac{a_n}{b_n}=\frac{e^{2n}+ne^n}{e^{2n}-n^2}$. After dividing both numerator and denominator by e^{2n}, we may write this as $\frac{a_n}{b_n}=\frac{1+\frac{n}{e^n}}{1-\frac{n^2}{e^{2n}}}$. Therefore, $\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{1+\frac{n}{e^n}}{1-\frac{n^2}{e^{2n}}}=1$. By the limit comparison test, we get what we want.

Example 4.15. The series $\sum_{n=1}^{\infty}\sin(\frac{1}{n^2})$ converges. When x is close to 0, $\sin(x)\approx x$. As $n\to\infty$, $\frac{1}{n^2}\to 0$ so we expect $\sin(\frac{1}{n^2})\approx\frac{1}{n^2}$. Since $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges, we expect that our original series does as well. Using $a_n = \sin(\frac{1}{n^2})$ and $b_n = \frac{1}{n^2}$, we see $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sin(\frac{1}{n^2})}{\frac{1}{n^2}} = \lim_{u\to 0} \frac{\sin(u)}{u} = 1$ via the substitution $u = \frac{1}{n^2}$. This says the series have the same behavior, so the result follows via the limit comparison test.

Example 4.16. Sometimes it's useful to chain comparison tests together. The series $\sum_{n=2}^{\infty} \frac{\ln(n)}{\sqrt{n^3-n^2}}$ is convergent. As $n \to \infty$, the fastest growing term in the denominator is n^3, so $\sqrt{n^3-n^2} \approx \sqrt{n^3} = n^{3/2}$. So we expect that $\sum_{n=2}^{\infty} \frac{\ln(n)}{\sqrt{n^3-n^2}}$ and $\sum_{n=2}^{\infty} \frac{\ln(n)}{n^{3/2}}$ have the same behavior. This can be checked using the limit comparison test: with $a_n = \frac{\ln(n)}{\sqrt{n^3-n^2}}$ and $b_n = \frac{\ln(n)}{n^{3/2}}$ we see that $\frac{a_n}{b_n} = \frac{n^{3/2}}{\sqrt{n^3-n^2}} \to 1$. Therefore, we just need to determine what $\sum_{n=2}^{\infty} \frac{\ln(n)}{n^{3/2}}$ does. In order to analyze this series, we need a fact about the growth speed of logarithms: they grow slower than any power function. Formalized mathematically, this says for any $a > 0$, there exists N such that $\ln(n) < n^a$ for $n \geq N$. Picking $a = 1/4$, this says $\ln(n) < n^{1/4}$ eventually, so that eventually $\sum \frac{\ln(n)}{n^{3/2}} < \sum \frac{n^{1/4}}{n^{3/2}} = \sum \frac{1}{n^{5/4}}$. The latter series is a convergent p-series, so by a direct comparison, $\sum \frac{\ln(n)}{n^{3/2}}$ converges and we are done.

5. The integral test

Conceptually, the integral test is the most important convergence test: it says that if the terms of an infinite series are "nice", the behavior of the series and the behavior of the corresponding improper integral should be the same. This provides the explicit link between infinite series and integration.

In practice, the integral test is often not that useful. In order for it to apply, you must know how to integrate the general term of an infinite series – this is something you either know you can do, in which case the test will work, otherwise if you don't know how to integrate the general term, the test is completely useless. It's generally best to try other convergence tests before trying the integral test, unless you are confident you can make it work.

Theorem 5.1 (**Integral test**). *Let $a_n = f(n)$ where $f(x)$ is a non-negative, continuous function that is eventually decreasing for $x \geq M$ for some M. Then $\sum a_n$ and $\int_M^{\infty} f(x)\,dx$ both converge or both diverge.*

Example 5.2. We can use the integral test to classify the behavior of p-series. Set $f(x) = \frac{1}{x^p}$. Then $f'(x) = -px^{p-1} < 0$ for $x > 0$, so that $f(x)$ is decreasing. It's clear that $f(x)$ is non-negative for $x > 0$, and also that it's continuous. Therefore by the integral test, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ and $\int_1^{\infty} \frac{1}{x^p}\,dx$ have the same behavior. First we handle the case $p \neq 1$. By definition, $\int_1^{\infty} \frac{1}{x^p}\,dx = \lim_{R\to\infty} \int_1^R x^{-p}\,dx = \lim_{R\to\infty} \frac{x^{1-p}}{1-p}\Big|_1^R = \lim_{R\to\infty} \frac{R^{1-p}}{1-p} - \frac{1}{1-p}$. If $p > 1$, then $1-p < 0$, so that $R^{1-p} \to 0$ as $R \to \infty$, so that $\int_1^{\infty} \frac{1}{x^p}\,dx = -\frac{1}{1-p} < \infty$. If $p < 1$, then $1-p > 0$, so that $R^{1-p} \to \infty$ as $R \to \infty$, which says $\int_1^{\infty} \frac{1}{x^p}\,dx$ diverges. In the case $p = 1$, the integral in question that we care about is $\int_1^{\infty} \frac{1}{x}\,dx = \lim_{R\to\infty} \int_1^R \frac{1}{x}\,dx = \lim_{R\to\infty} \ln|x|\Big|_1^R = \lim_{R\to\infty} \ln(R) = \infty$, so that $\int_1^{\infty} \frac{1}{x}\,dx$ diverges. Putting this all together, we find that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for $p > 1$ and diverges for $p \leq 1$.

Example 5.3. One place where the integral test really shines is when there are logarithms floating around: the series $\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$ diverges. Set $f(x) = \frac{1}{x\ln(x)}$. Then $f'(x) =$

$-\frac{(1+\ln(x))}{x^2\ln(x)^2} < 0$ for $x \geq 2$. The function $f(x)$ is also non-negative for $x \geq 2$, and it's clearly continuous, so by the integral test, the series $\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$ and the improper integral $\int_2^{\infty} \frac{1}{x\ln(x)}\,dx$ have the same behavior. By definition, the latter integral is $\lim_{R\to\infty} \int_2^R \frac{1}{x\ln(x)}\,dx$ $= \lim_{R\to\infty} \ln(\ln(x))\Big|_2^R = \lim_{R\to\infty} \ln(\ln(R)) - \ln(\ln(2)) = \infty$. Therefore, the integral diverges, so that the series diverges.

The right way to really think about the integral test is not as a test for convergence of infinite series, but as a test for convergence of *integrals*. The integral test is incredibly important if you think about it this way, because it gives us significantly more techniques than we had before to determine if integrals converge or diverge!

Example 5.4. Suppose we wanted to know if $\int_0^{\infty} \frac{x^3+x+1}{x^4+1}\,dx$ converges or diverges. Using what we learned before, we could split the integral up and do several direct comparison tests. Alternatively, we could use the integral test, and then a limit comparison test. The function $f(x) = \frac{x^3+x+1}{x^4+1}$ is clearly non-negative and continuous, and $f'(x) = -\frac{(x^6+3x^4+4x^3-3x^2-1)}{(x^4+1)^2}$ is negative for $x > 1$ (which is not terribly hard to see). Therefore, by the integral test, $\int_0^{\infty} \frac{x^3+x+1}{x^4+1}\,dx$ and $\sum_{n=0}^{\infty} \frac{n^3+n+1}{n^4+1}$ have the same behavior. As $n \to \infty$, $\frac{n^3+n+1}{n^4+1} \approx \frac{1}{n}$. Doing a limit comparison test on $\sum_{n=0}^{\infty} \frac{n^3+n+1}{n^4+1}$ with $\sum_{n=1}^{\infty} \frac{1}{n}$ will show it diverges, and so the integral diverges as well.

The proof of the integral test gives the following upper and lower bounds of the sum:

Theorem 5.5 (Integral test estimate)**.** *Suppose that $\sum_{n=0}^{\infty} a_n$ is a convergent series satisfying the conditions of the integral test with a_n monotonically decreasing for $n \geq N$. Then $\int_N^{\infty} f(x)\,dx \leq \sum_{n=N}^{\infty} a_n \leq a_N + \int_N^{\infty} f(x)\,dx$.*

Example 5.6. Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent series satisfying the conditions of the integral test, we have $1 = \int_1^{\infty} \frac{1}{x^2}\,dx \leq \sum_{n=1}^{\infty} \frac{1}{n^2} \leq 1 + \int_1^{\infty} \frac{1}{x^2}\,dx = 2$. The actual value of the sum was shown by Euler to be $\frac{\pi^2}{6} \approx 1.645$!

6. The root and ratio tests

We now move on to series tests that are applicable to terms with negative terms. So far, none of our convergence tests have been "easy", in the sense that given a series, we can't just test if it converges or diverges by itself. We fix this with the root and ratio tests, which are arguably the easiest to use convergence tests. Before we do that, we need some terminology that applies to series that have negative terms.

Definition 6.1. An infinite series $\sum a_n$ **converges absolutely** if $\sum |a_n|$ converges. If $\sum |a_n|$ diverges and $\sum a_n$ converges, then we say $\sum a_n$ **converges conditionally**.

Absolute convergence is a "stronger" form of convergence, in the following sense:

Theorem 6.2 (**Absolute convergence test**)**.** *If $\sum |a_n|$ converges, then $\sum a_n$ converges. That is, an absolutely convergent series converges.*

Example 6.3. We'll see later that $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ converges while $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, so that $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ is a conditionally convergent series.

Example 6.4. The series $\sum_{n=1}^{\infty} \frac{\cos(n)}{2^n}$ converges absolutely. Taking absolute values, since $|\cos(n)| \leq 1$, we have $\sum_{n=1}^{\infty} \frac{|\cos(n)|}{2^n} \leq \sum_{n=1}^{\infty} \frac{1}{2^n}$ which is a convergent geometric series.

We now state the ratio and root tests:

Theorem 6.5 (**Ratio test**). *Let $\sum a_n$ be an infinite series. Set $L = \lim_{n\to\infty} |\frac{a_{n+1}}{a_n}|$.*
- (a) *If $0 \leq L < 1$, then $\sum a_n$ converges absolutely.*
- (b) *If $L > 1$, then $\sum a_n$ diverges.*
- (c) *if $L = 1$, the ratio test says nothing.*

Theorem 6.6 (**Root test**). *Let $\sum a_n$ be an infinite series. Set $L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$.*
- (a) *If $0 \leq L < 1$, then $\sum a_n$ converges absolutely.*
- (b) *If $L > 1$, then $\sum a_n$ diverges.*
- (c) *if $L = 1$, the root test says nothing.*

Both the root and ratio test require only a single infinite series to perform the test, and further more, when they work they tell you explicitly whether or not the series converges (even absolutely!) or diverges. The drawback is that they don't always work. Of the two, the ratio test is more useful in practice. In fact, the root test is really only helpful when there are expressions raised to n-th powers, and *don't* involve any factorials. The ratio test is significantly more useful when factorials appear (more than any other convergence test), and handles n-th powers relatively easily as well. If you're trying to determine if an infinite series converges or not, and it's not of a special form (i.e. alternating, geometric, telescoping), and there's nothing to obviously do a limit comparison with, I recommend trying the ratio test.

One last thing that's worth pointing out: it's not a coincidence that the two tests look very similar. The root test is actually *stronger* than the ratio test, in the sense that you prove the ratio test by using the root test. If the ratio test works, you could have also done the root test. Sometimes if the ratio test doesn't work, the root test will work. If the root test *doesn't* work, don't bother with the ratio test: it won't work either!

Example 6.7. The series $\sum_{n=1}^{\infty} \frac{2n}{n^n}$ converges. Set $a_n = \frac{2n}{n^n}$. Then $\sqrt[n]{|a_n|} = \frac{(2n)^{1/n}}{n}$. As $n \to \infty$, we see that $(2n)^{1/n} \to 1$: this is because if $L = \lim_{n\to\infty}(2n)^{1/n}$, then $\ln(L) = \lim_{n\to\infty} \frac{\ln(2n)}{n} = 0$ by L'Hopital's rule, so $L = 1$. This says $\sqrt[n]{|a_n|} \to 0$ as $n \to \infty$. Convergence then follows from the root test.

Example 6.8. The series $\sum_{n=1}^{\infty}(1 + \frac{1}{n})^{-n^2}$ converges. Set $a_n = (1 + \frac{1}{n})^{-n^2}$, then $\sqrt[n]{|a_n|} = (1 + \frac{1}{n})^{-n}$. As $n \to \infty$, we see $\lim_{n\to\infty} \frac{1}{(1+1/n)^n} = \frac{1}{e} < 1$, so the convergence follows from the root test.

Example 6.9. The series $\sum_{n=1}^{\infty} \frac{n!}{(2n)!}$ converges. Set $a_n = \frac{n!}{(2n)!}$. Then $|\frac{a_{n+1}}{a_n}| = \frac{(n+1)!}{(2n+2)!} \cdot \frac{(2n)!}{n!} = \frac{n+1}{(2n+2)(2n+1)} \to 0 < 1$ as $n \to \infty$. The convergence then follows by the ratio test.

Example 6.10. The series $\sum_{n=1}^{\infty} \frac{2^{n^2}}{n!}$ diverges. Set $a_n = \frac{2^{n^2}}{n!}$, then $|\frac{a_{n+1}}{a_n}| = \frac{2^{n^2+2n+1}}{(n+1)!} \cdot \frac{n!}{2^{n^2}} = \frac{2^{2n+1}}{n+1} \to \infty$. Therefore the series diverges by the ratio test.

Example 6.11. The series $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ converges. Set $a_n = \frac{n!}{n^n}$. Then $|\frac{a_{n+1}}{a_n}| = \frac{(n+1)!n^n}{(n+1)^{n+1}n!} = \frac{(n+1)n^n}{(n+1)(n+1)^n} = (\frac{n}{n+1})^n = (1 + \frac{1}{n})^{-n}$. As $n \to \infty$, we have $(1 + \frac{1}{n})^{-n} \to \frac{1}{e}$, so the result follows by the ratio test.

7. Alternating series

Our last type of series we study is when the negative terms are predictable, specifically, when the terms of the series alternate between positive and negative.

Definition 7.1. An **alternating series** is an infinite series of the form $\sum(-1)^n a_n$ where $a_n \geq 0$.

Because of the alternation between positive and negative terms, this makes it harder for the sum to diverge to infinity, so in some sense, alternating series are more "well behaved". One way of phrasing this is as follows:

Theorem 7.2 (**Alternating series test**). *Let* $\sum(-1)^n a_n$ *be an alternating series. If* $\lim_{n\to\infty} a_n = 0$ *and* a_n *is monotonically decreasing, then* $\sum(-1)^n a_n$ *converges.*

The first condition of the alternating series test is a requirement for the series to even converge in the first place (otherwise it diverges by the divergence test), so really the only condition we are imposing on the terms is that they are strictly decreasing, which is a relatively tame condition as far as "niceness" of sequences go. The alternating series test has one weakness: it cannot show an alternating series diverges. In fact, the conditions of the alternating series test say that a divergent alternating series has to be fairly complicated (provided it doesn't obviously diverge, i.e. has $\lim_{n\to\infty} a_n \neq 0$).

Example 7.3. The series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ converges conditionally: we know $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, but clearly $\frac{1}{n}$ goes to 0 and is monotonically decreasing, and so by the alternating series test, $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ converges conditionally.

Example 7.4. The series $\sum_{n=1}^{\infty}(-1)^n \frac{1}{1+\frac{1}{n}}$ diverges because $\lim_{n\to\infty}(-1)^n \frac{1}{1+\frac{1}{n}}$ does not exist – not every alternating series requires the alternating series test!

Example 7.5. The series $\sum_{n=2}^{\infty}(-1)^n \frac{1}{n^2 \ln(n)}$ converges. With $a_n = \frac{1}{n^2 \ln(n)}$, it's clear that $\lim_{n\to\infty} a_n = 0$, and we see that a_n is decreasing because the function $f(n) = \frac{1}{n^2 \ln(n)}$ has derivative $f'(n) = -\frac{2\ln(n)+1}{n^3 \ln(n)^2} < 0$ for $n \geq 1$. The result follows by the alternating series test. In fact, the convergence is absolute: the series $\sum_{n=2}^{\infty} \frac{1}{n^2 \ln(n)}$ converges because $n^2 \ln(n) > n^2$ for $n \geq 3$, so $\sum_{n=3}^{\infty} \frac{1}{n^2 \ln(n)} < \sum_{n=3}^{\infty} \frac{1}{n^2}$ which is a convergent p-series.

Example 7.6. The series $\sum_{n=1}^{\infty} \frac{\cos(\pi n)}{n^{2/3}}$ converges conditionally. Notice that $\cos(\pi n) = (-1)^n$, so this is really just the series $\sum_{n=1}^{\infty}(-1)^n \frac{1}{n^{2/3}}$. It's clear that $\frac{1}{n^{2/3}} \to 0$ as $n \to \infty$, and the function $f(n) = n^{-2/3}$ has derivative $f'(n) = -\frac{2}{3}n^{-5/3} < 0$, so it converges by the alternating series test. However, taking an absolute value, the series $\sum_{n=1}^{\infty} \frac{1}{n^{2/3}}$ is a divergent p-series.

Example 7.7. The series $\sum_{n=1}^{\infty}(-1)^n \frac{e^{1/n}}{n}$ converges conditionally. As $n \to \infty$, $\frac{1}{n} \to 0$, and $e^x \approx 1$ for $x \approx 0$. This says as $n \to \infty$, that $\frac{e^{1/n}}{n} \approx \frac{1}{n}$. First, we show that we do not converge absolutely: the series $\sum_{n=1}^{\infty} \frac{e^{1/n}}{n}$ should behave like the series $\sum_{n=1}^{\infty} \frac{1}{n}$, which diverges. Indeed, using the limit comparison test with $a_n = \frac{e^{1/n}}{n}$ and $b_n = \frac{1}{n}$, we have $\frac{a_n}{b_n} = e^{1/n}$ and clearly $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$. By the limit comparison test, $\sum_{n=1}^{\infty} \frac{e^{1/n}}{n}$ diverges. The alternating series however, converges. With $a_n = \frac{e^{1/n}}{n}$, it's clear that $a_n \geq 0$ and

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{e^{1/n}}{n} = 0$. Set $f(n) = \frac{e^{1/n}}{n}$. Then $f'(n) = -\frac{e^{1/n}(n+1)}{n^3} < 0$. This says a_n is decreasing, so by the alternating series test, we are done.

Example 7.8. The series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{10^n}{2^{n^2}}$ converges absolutely. To see this, use the root test: $|a_n|^{1/n} = \frac{10}{2^n}$ and clearly $|a_n|^{1/n} \to 0$ as $n \to \infty$. The alternating series test isn't the only thing you should try when you see alternating series!

Example 7.9. We give an example of a (non-obvious) divergent alternating series. Define

$$a_n = \begin{cases} 1/n & n \text{ is even} \\ 1/n^2 & n \text{ is odd} \end{cases}$$

I claim the series $\sum_{n=1}^{\infty}(-1)^n a_n$ diverges. We show this via the definition of convergence for an infinite series. The N-th partial sum is given by $S_N = \sum_{n=1}^{N}(-1)^n a_n = -\sum_{k\leq N,\text{ k odd}} \frac{1}{k^2} + \sum_{k\leq N,\text{ k even}} \frac{1}{k}$. As $N \to \infty$, the first sum converges, because $\sum_{k \text{ odd}} \frac{1}{k^2} \leq \sum_{n=1}^{\infty} \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, while the second sum diverges, because $\sum_{k \text{ even}} \frac{1}{k} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n}$ is the harmonic series. Since the partial sums diverge, the series $\sum_{n=1}^{\infty}(-1)^n a_n$ diverges.

Alternating series have a very nice error bound, that make it very easy to estimate these types of sums.

Theorem 7.10 (Alternating series error bound)**.** *Let $\sum_{n=0}^{\infty}(-1)^n a_n$ be a convergent alternating series, and let S denote the value of the sum. Then $|S - S_N| \leq a_{N+1}$.*

Example 7.11. We saw that $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges. For any value of N, $\sum_{n=1}^{N} \frac{(-1)^n}{n}$ approximates the true value of the sum within an error of $\frac{1}{N+1}$. To guarantee two decimal places of accuracy, we can take $N = 99$, for example, and one may compute with a computer that $\sum_{n=1}^{99} \frac{(-1)^n}{n} \approx -.698$, so that $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \approx -.69$. We'll see later that $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = -\ln(2) \approx -.693147$ – the series converges very slowly!

8. Summary of tests

Test	Applicable Series	Conclusion	Additional
Divergence	$\sum a_n$	Diverges if $\lim_{n\to\infty} a_n \neq 0$	Always try this first. Inconclusive if $\lim_{n\to\infty} a_n = 0$. Can **not** show convergence!!
Geometric Series	$\sum_{n=M}^{\infty} cr^n$	Converges if $\lvert r\rvert < 1$, diverges if $\lvert r\rvert \geq 1$	Converges to value $\dfrac{cr^M}{1-r}$
Direct Comparison	$\sum a_n$ and $\sum b_n$ with $0 \leq a_n \leq b_n$ eventually	If $\sum b_n$ converges, then $\sum a_n$ converges If $\sum a_n$ diverges, then $\sum b_n$ diverges	
Limit Comparison	$\sum a_n$ and $\sum b_n$ with $0 < a_n, b_n$ and $\lim_{n\to\infty} \dfrac{a_n}{b_n} = L$, $0 < L < \infty$	$\sum a_n$ and $\sum b_n$ both converge or diverge	
Integral	$\sum a_n$ with $a_n = f(n)$ continuous, positive, decreasing eventually for $n \geq M$	$\sum a_n$ and $\int_M^\infty f(x)\,dx$ both converge or diverge	$\lvert S - S_N\rvert \leq \int_N^\infty f(x)\,dx$
p-Series	$\sum_{n=1}^{\infty} \dfrac{1}{n^p}$	Converges if $p > 1$, diverges if $p \leq 1$	
Absolute Convergence	$\sum a_n$	If $\sum \lvert a_n\rvert$ converges, $\sum a_n$ converges absolutely	If $\sum a_n$ converges but $\sum \lvert a_n\rvert$ diverges, we call this conditional convergence
Ratio	$\sum a_n$ with $a_n \neq 0$ and $\lim_{n\to\infty} \left\lvert \dfrac{a_{n+1}}{a_n} \right\rvert = L$	Converges (absolutely) if $L < 1$, diverges if $L > 1$	Inconclusive if $L = 1$
Root	$\sum a_n$ with $\lim_{n\to\infty} \sqrt[n]{\lvert a_n\rvert} = L$	Converges (absolutely) if $L < 1$, diverges if $L > 1$	Inconclusive if $L = 1$
Alternating Series	$\sum (-1)^n a_n$ with a_n positive, monotonically decreasing eventually, and $\lim_{n\to\infty} a_n = 0$	$\sum (-1)^n a_n$ converges	$\lvert S - S_N\rvert \leq a_{N+1}$