POWER SERIES

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These notes are a treatment of the theory of power series as covered in a second semester calculus course like Math 31B at UCLA. There may be many typos – please let me know if any are found!

1. introduction

Our study of when infinite series converge leads to the following question: when can a function be written as an infinite series? To motivate why one would even want to do such a thing, at its heart, calculus is about *approximation*. One of the questions calculus tries to answer is how to find "good" approximations to a function $f(x)$ locally near some point x_0. You are already familiar with one such technique: "linearize" the function by finding the tangent line $L(x)$ at x_0, and then for values of x close to x_0, $L(x)$ is a "good" approximation to $f(x)$. $L(x)$ is an approximation to $f(x)$ by a polynomial of degree 1. What if we wanted an approximation to $f(x)$ by a polynomial of degree 2, or arbitrary degree n?

Since calculus is concerned with limit operations, one might ask if we could make sense of an "infinite degree" polynomial. If so, it's then natural to ask whether or not the above approximations actually become equalities, which is precisely asking when can you write a function as an infinite series! Our goal is to answer this question, and see the many powerful application that this knowledge gives us with regards to classical problems in calculus.

2. Basic definitions and examples

Definition 2.1. A **power series** is an infinite series of the form $F(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$ for variable x, some sequence $\{a_n\}$, and some real number c, called the **center** of the power series.

Note that a power series $F(x)$ is not necessarily a well-defined function: for some values of x, the resulting series $F(x)$ may either converge or diverge.

Example 2.2. Let $F(x) = \sum_{n=0}^{\infty} x^n$, which is a power series centered at $c = 0$ with constant coefficients $a_n = 1$. For each fixed value of x, the resulting infinite series is a geometric series, and therefore converges if $|x| < 1$ and diverges if $|x| \geq 1$. Therefore we cannot make sense of $F(x)$ as a function defined on $\mathbb{R}$, but we *can* make sense of $F(x)$ as a function defined on $(-1, 1)$: from the formula for the sum of a geometric series, we know that for $|x| < 1$, $F(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$.

The first question we must answer then if we wish to make sense of power series, is when can we determine a domain that makes a power series a well-defined function? Since for each fixed value of x a power series is just an infinite series, we can answer this using the theory we've already developed. A bit of work will show that power series have the following behavior:

Theorem 2.3 (Convergence of power series)**.** *For a power series $F(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$, exactly one of the following is true:*

- *There is a unique non-negative real number R such that $F(x)$ converges absolutely for $|x-c| < R$ and diverges for $|x-c| > R$.*
- *$F(x)$ converges absolutely for all $x \in \mathbb{R}$.*

Definition 2.4. The **radius of convergence** R of a power series $F(x)$ is defined as the number R in the above theorem. If $F(x)$ converges absolutely for all x, we define $R = \infty$. The **interval of convergence** is the set of all values such that $F(x)$ converges.

How can we find the radius of convergence of a power series? Assuming we can apply the ratio test to the infinite series $\sum_{n=0}^{\infty} a_n(x-c)^n$, we see that the series converges absolutely if $\lim_{n\to\infty} |\frac{a_{n+1}(x-c)^{n+1}}{a_n(x-c)^n}| = \lim_{n\to\infty} |\frac{a_{n+1}}{a_n}||x-c| < 1$, and diverges if $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}||x-c| > 1$. If $L = \lim_{n\to\infty} |\frac{a_{n+1}}{a_n}|$ is finite and non-zero, this says that the infinite series $F(x)$ converges absolutely if $L|x-c| < 1$ and diverges if $L|x-c| > 1$, i.e. converges absolutely for $|x-c| < 1/L$ and diverges if $|x-c| > 1/L$, so that the above theorem says $R = 1/L$. If $L = 0$, then $L|x-c| = 0$ for any value of x, so therefore $F(x)$ converges absolutely for any such choice of x, which says $R = \infty$. If L is infinite, then for any value of $x \neq c$, $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}||x-c| = \infty$, so $F(x)$ diverges, and for $x = c$ we see $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}||x-c| = 0$, i.e. $F(x)$ converges only at $x = c$, so $R = 0$. We can sum this up in the following:

Theorem 2.5. *Assume that $L = \lim_{n\to\infty} |\frac{a_{n+1}}{a_n}|$ exists. The radius of convergence of the power series $F(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$ is given by $R = \frac{1}{L}$, where this is interpreted as $R = 0$ if $L = \infty$ or $R = \infty$ if $L = 0$.*

The theorem on the convergence behavior of power series tells us that if $R < \infty$, a power series must converge in the interval $(c-R, c+R)$, and diverges in $(-\infty, c-R) \cup (c+R, \infty)$. However, the theorem tells us nothing about what happens at the endpoints $x = c-R$ and $x = c+R$. To check if a power series converges for these values of x, this must be done manually using the usual convergence tests for infinite series.

Example 2.6. In the previous example, we determined the power series $F(x) = \sum_{n=0}^{\infty} x^n$ converges if $|x| < 1$ and diverges if $|x| \geq 1$ using properties of geometric series. In other words, the radius of convergence is $R = 1$ and the interval of convergence is $(-1, 1)$. We can also determine this using the ratio test: $F(x)$ converges absolutely if $\lim_{n\to\infty} |\frac{x^{n+1}}{x^n}| = |x| < 1$ and diverges if $|x| > 1$, so $R = 1$. If $x = 1$, then $F(1) = \sum_{n=0}^{\infty} 1$ diverges, and similarly $F(-1) = \sum_{n=0}^{\infty} (-1)^n$ also diverges, so the interval of convergence is $(-1, 1)$.

Example 2.7. Set $F(x) = \sum_{n=0}^{\infty} n!x^n$. What is the interval of convergence of $F(x)$? Using the ratio test, we find that $L = \lim_{n\to\infty} \frac{(n+1)!}{n!} = \lim_{n\to\infty}(n+1) = \infty$. This says $R = 0$, and so $F(x)$ converges only at $x = 0$.

Example 2.8. Set $F(x) = \sum_{n=2}^{\infty} \frac{1}{\ln(n)}(x-1)^n$. What is the interval of convergence of $F(x)$? Using the ratio test, $F(x)$ converges absolutely if $\lim_{n\to\infty} |\frac{\frac{1}{\ln(n+1)}(x-1)^{n+1}}{\frac{1}{\ln(n)}(x-1)^n}| = \lim_{n\to\infty} \frac{\ln(n)}{\ln(n+1)}|x-1| < 1$ and diverges if $\lim_{n\to\infty} \frac{\ln(n)}{\ln(n+1)}|x-1| > 1$. Since $\lim_{n\to\infty} \frac{\ln(n)}{\ln(n+1)} = 1$, this says $F(x)$ converges absolutely if $|x-1| < 1$ and diverges if $|x-1| > 1$, i.e. $R = 1$. We then see that $F(x)$ converges in the interval $(0, 2)$. What happens at the endpoints? At $x = 0$, we have $F(0) = \sum_{n=2}^{\infty} \frac{1}{\ln(n)}(-1)^n$, which converges by the alternating series test. At $x = 2$,

$F(1) = \sum_{n=2}^{\infty} \frac{1}{\ln(n)}$. Since $\ln(n) < n$ for all n, $\frac{1}{n} < \frac{1}{\ln(n)}$ so that $\sum_{n=2}^{\infty} \frac{1}{\ln(n)}$ diverges by a direct comparison. Therefore $F(1)$ diverges, and the interval of convergence of $F(x)$ is given by $[0, 2)$.

Example 2.9. Set $F(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{4^n (n!)^2}$. In both of the two previous examples, we could have computed the radius of convergence by using the previous theorem. However here the theorem does not apply, because the power series $F(x)$ has only *even* powers of x in the sum. Therefore we need to use the ratio test to determine the radius of convergence. With $b_n = \frac{(-1)^n x^{2n}}{4^n (n!)^2}$, we have $|\frac{b_{n+1}}{b_n}| = |\frac{(-1)^{n+1} x^{2n+2}}{4^{n+1}((n+1)!)^2} \cdot \frac{4^n (n!)^2}{(-1)^n x^{2n}}| = \frac{x^2}{4(n+1)^2}$. Therefore, $\lim_{n\to\infty} |\frac{b_{n+1}}{b_n}| = \lim_{n\to\infty} \frac{x^2}{4(n+1)^2} = 0$ for any value of x. This says $R = \infty$ and $F(x)$ converges absolutely for all x.

3. Functions defined by power series

The convergence behavior of power series says that a power series $F(x)$ determines a well-defined function on its interval of convergence. One reason why we care about power series is that doing calculus with them is extremely easy:

Theorem 3.1 (Integration and differentiation of power series)**.** *Let $F(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$ be a power series with radius of convergence R. Then for $|x - c| < R$, we may differentiate and integrate the power series $F(x)$ term by term. That is, the following hold:*

- $F'(x) = \sum_{n=0}^{\infty} \frac{d}{dx} a_n (x-c)^n = \sum_{n=1}^{\infty} n a_n (x-c)^{n-1}$
- $\int F(x)\, dx = \sum_{n=0}^{\infty} \int a_n (x-c)^n\, dx = C + \sum_{n=0}^{\infty} \frac{a_n}{n+1} (x-c)^{n+1}$.

Furthermore, the radius of convergence remains unchanged, but the interval of convergence of these new series may differ at the endpoints.

Example 3.2. Let $F(x) = \sum_{n=1}^{\infty} \frac{x^n}{n}$. Then using the theorem for computing the radius of convergence, we see that $R = 1$ and $F(x)$ converges in $(-1, 1)$. If $x = 1$, $F(1) = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges, and at $x = -1$, $F(-1) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is a convergent alternating series. Therefore, $F(x)$ has interval of convergence $[-1, 1)$. Taking a derivative says $F'(x) = \sum_{n=1}^{\infty} \frac{d}{dx} \frac{x^n}{n} = \sum_{n=1}^{\infty} x^{n-1} = \sum_{n=0}^{\infty} x^n$. As previously determined, this power series has radius of convergence 1 and interval of convergence $(-1, 1)$. If we integrate $F(x)$, we see $\int F(x)\, dx = \sum_{n=1}^{\infty} \int \frac{x^n}{n}\, dx = \sum_{n=1}^{\infty} \frac{1}{n} \int x^n\, dx = C + \sum_{n=1}^{\infty} \frac{1}{n(n+1)} x^{n+1}$. This series has radius of convergence $R = 1$, and so converges in the interval $(-1, 1)$. At $x = -1$, the series $C + \sum_{n=1}^{\infty} \frac{(-1)^n}{n(n+1)}$ is a convergent alternating series, and at $x = 1$ the series $C + \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges by a limit comparison with $\sum_{n=1}^{\infty} \frac{1}{n^2}$. This says the interval of convergence is $[-1, 1]$.

The above theorem tells us that functions defined by a power series are very special: they are not only differentiable, but are *infinitely* differentiable (the derivative of a power series is a power series so you can keep applying the theorem!), and they behave nicely with respect to the operations of differentiation and integration. Naturally then, is the following: give a function $f(x)$, how can we determine if it can be defined by a power series on some interval?

4. Taylor Series

Suppose we have a function $f(x)$ defined on some interval I that can be written as a power series. That is to say, $f : I \to \mathbb{R}$ is defined by $f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$ for some power series $\sum_{n=0}^{\infty} a_n(x-c)^n$. It turns out the coefficients a_n of the power series can very easily be determined. Plugging in $x=c$, all terms on the right hand side disappear except the $n=0$ term, so $f(c)=a_0$. Since f is represented by a power series, it is differentiable, so we may write $f'(x)=\sum_{n=1}^{\infty} na_n(x-c)^{n-1}$. Plugging in $x=c$, all terms in the right hand side disappear except the $n=1$ term, which says $f'(c)=a_1$. Differentiating again says $f''(x)=\sum_{n=2}^{\infty} n(n-1)a_n(x-c)^{n-2}$. Plugging in $x=c$, all terms in the right hand side disappear except the $n=2$ term, so $f''(c)=2a_2$ says $a_2=\frac{f''(c)}{2}$. Continuing this process, one finds that $a_n=\frac{f^{(n)}(c)}{n!}$, so that $f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(c)}{n!}(x-c)^n$. This says that if a function $f(x)$ can be written as a power series, it necessarily has this special form.

Definition 4.1. Let $f(x)$ be an infinitely differentiable function. The **Taylor series** of $f(x)$ centered at c, denoted $T(x)$, is the power series $T(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(c)}{n!}(x-c)^n$. If $c=0$, the power series is sometimes called the **MacLaurin series** of $f(x)$.

Example 4.2. What's the Taylor series of $f(x)=e^x$ centered at $c=0$? By definition, this Taylor series is given by $\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n$, so we need to figure out what an arbitrary n-th order derivative of f looks like. Luckily, $f^{(n)}(x)=e^x$ for all n, so $f^{(n)}(0)=1$. This says the Taylor series centered at 0 of e^x is given by $\sum_{n=0}^{\infty}\frac{1}{n!}x^n$.

Example 4.3. We have seen that a valid power series expansion of $f(x)=\frac{1}{1-x}$ when $|x|<1$. By the uniqueness of a power series representation, this actually says that the Taylor series centered at $c=0$ of $f(x)$ is given by $\sum_{n=0}^{\infty} x^n$, valid for $|x|<1$. That is, $\frac{1}{1-x}=\sum_{n=0}^{\infty} x^n$ for $|x|<1$.

Example 4.4. What's the Taylor series centered at $c=0$ for $f(x)=\sin(x)$? Derivatives of $\sin(x)$ have a simple pattern: they cycle $\cos(x),-\sin(x),-\cos(x),\sin(x)$. If we plug in $x=0$, the pattern goes $1,0,-1,0$, i.e. the even order derivatives at 0 are all 0 and the odd order derivatives at 0 alternate between 1 and -1. Then $\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n=\sum_{n \text{ odd}}\frac{f^{(n)}(0)}{n!}x^n$. We can loop the sum over all odd integers by writing $n=2k+1$ and then letting k vary from 0 to ∞, i.e. $\sum_{n \text{ odd}}\frac{f^{(n)}(0)}{n!}x^n=\sum_{k=0}^{\infty}\frac{f^{(2k+1)}(0)}{(2k+1)!}x^{2k+1}=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}x^{2k+1}$.

What we determined at the beginning of the section is that *if* a function can be written as a power series centered at some point c, that power series *must* be its Taylor series. We have **not** said that a function is equal to its Taylor series. Indeed, this is false:

Example 4.5. Consider the function f defined by $f(x)=\begin{cases} e^{-1/x^2} & x\neq 0 \\ 0 & x=0 \end{cases}$. Through quite a lot of (difficult) work, one can show that this function has the following unusual properties: f is infinitely differentiable, and $f^{(n)}(0)=0$ for all $n\geq 0$. The Taylor series of $f(x)$ centered at $c=0$ is then given by $T(x)=0$, which obviously is not the same as $f(x)$.

To finish off the section, we give some examples of how one goes about computing Taylor series. The take away from these examples should all be the same: to compute a Taylor series, perform operations on *known* power series to arrive at an answer. Do not try and work with the definition!

Example 4.6. Let's compute the Taylor series of $f(x)=\frac{x^2}{(1-x)^3}$ centered at $c=0$. We start with the known Taylor series $\frac{1}{1-x}=\sum_{n=0}^{\infty} x^n$, valid for $|x|<1$. If we differentiate once, we find $\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1}$, and if we differentiate again we see $\frac{2}{(1-x)^3} = \sum_{n=2}^{\infty} n(n-1)x^{n-2}$. This says $\frac{1}{(1-x)^3} = \frac{1}{2}\sum_{n=2}^{\infty} n(n-1)x^{n-2}$, so multiplying by x^2 says $\frac{x^2}{(1-x)^3} = \frac{x^2}{2}\sum_{n=2}^{\infty} n(n-1)x^{n-2} = \frac{1}{2}\sum_{n=2}^{\infty} n(n-1)x^n$. This expansion is still valid for $|x| < 1$ (differentiating doesn't change the radius of convergence!). Since we have found a power series representing $f(x)$ in some interval, this *forces* it to be the Taylor series of $f(x)$ by uniqueness.

Example 4.7. Similarly, by integrating the Taylor series of $\frac{1}{1-x}$, we can find the Taylor series of $\ln(1-x)$. We have $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, valid for $|x| < 1$. Integrating says $\ln(1-x) = C + \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$. To figure out C, plug in $x = 0$: we then have $\ln(1) = C$, so $C = 0$. It's easy to see we pick up convergence at the end point $x = -1$, so we have $\ln(1-x) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$ for $x \in [-1, 1)$. Since we have found a power series representation for our function, it must be its Taylor series.

Example 4.8. Let's compute the Taylor series of $f(x) = \frac{1}{1-x}$ centered at $c = 4$. We know that $\frac{1}{1-u} = \sum_{n=0}^{\infty} u^n$ is a valid power series expansion when $|u| < 1$. Instead of directly computing derivatives, we can do a clever trick to find the Taylor series. We know that the Taylor series of $f(x)$ centered at 4 is of the form $\sum_{n=0}^{\infty} a_n(x-4)^n$ for some coefficients a_n, so if we can find such a power series, uniqueness forces it to be the Taylor series of $f(x)$. To do so, we will perform a substitution and use the above formula to make an $(x-4)^n$ term appear in the sum. Write $\frac{1}{1-x} = \frac{1}{1-(x-4+4)} = \frac{1}{-3-(x-4)} = -\frac{1}{3}\frac{1}{1-(-\frac{x-4}{3})}$. Set $u = -\frac{x-4}{3}$. Then the above says $\frac{1}{1-x} = -\frac{1}{3}\sum_{n=0}^{\infty}(-\frac{x-4}{3})^n = \sum_{n=0}^{\infty}(-1)^{n+1}\frac{(x-4)^n}{3^{n+1}}$, which is valid for $|\frac{x-4}{3}| < 1$, i.e. $|x-4| < 3$. Since we have found a power series of f centered at 4, it must be its Taylor series.

Example 4.9. Let's find the Taylor series of $f(x) = \frac{2}{1-2x} - \frac{1}{1-x}$ centered at $c = 0$. Similarly to above, we use the expansion $\frac{1}{1-u} = \sum_{n=0}^{\infty} u^n$ for $|u| < 1$. Writing each term as a power series, we have $\frac{2}{1-2x} - \frac{1}{1-x} = 2\sum_{n=0}^{\infty}(2x)^n - \sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} 2^{n+1}x^n - \sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty}(2^{n+1}-1)x^n$, which is only valid when *both* series converge. Since the first series convergences only for $|x| < 1/2$, we see this power series expansion is only valid for $|x| < 1/2$. Since we have found a power series expansion of f that's valid in some interval around 0, this says it must be the Taylor series of f.

Example 4.10. Let's find the Taylor series of $f(x) = \int_0^x \frac{e^{t^2}-1}{t}\,dt$ centered at $c = 0$. Note that it is not at all possible to compute an anti-derivative of the integrand – the only method here is to integrate its Taylor series. The Taylor series of e^t centered at 0 is given by $e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!}$ which is valid for all t, so $e^{t^2} = \sum_{n=0}^{\infty} \frac{(t^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{t^{2n}}{n!}$. This series looks like $1 + t^2 + \frac{t^4}{2} + \ldots$, so $e^{t^2} - 1 = t^2 + \frac{t^4}{2} + \ldots = \sum_{n=1}^{\infty} \frac{t^{2n}}{n!}$, which is a valid expansion for all t. Dividing through by t then says $\frac{e^{t^2}-1}{t} = \frac{1}{t}\sum_{n=1}^{\infty} \frac{t^{2n}}{n!} = \sum_{n=1}^{\infty} \frac{t^{2n-1}}{n!}$. We then have $\int_0^x \frac{e^{t^2}-1}{t}\,dt = \int_0^x \sum_{n=1}^{\infty} \frac{t^{2n-1}}{n!}\,dt = \sum_{n=1}^{\infty} \int_0^x \frac{t^{2n-1}}{n!}\,dt = \sum_{n=1}^{\infty} \frac{x^{2n}}{(2n)n!}$, and further this expression is valid for all x. Since we have found a power series representation of $f(x)$, this must be its Taylor series.

Example 4.11. Let's find the Taylor series of $f(x) = \tan^{-1}(x)$ centered at $c = 0$. We know that $\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2}$, so let's start by finding the Taylor series of this function instead, which is much easier. Starting with $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, we have $\frac{1}{1+x^2} = \sum_{n=0}^{\infty}(-1)^n x^{2n}$

by replacing x with $-x^2$. Integrating then says $\tan^{-1}(x) = C + \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}$. Since $\tan^{-1}(0) = 0$, we find $C = 0$, and so $\tan^{-1}(x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}$. The radius of convergence is 1 because we did not do any operations to change it from our starting series. Testing the endpoints, the series converges at both $x = 1, -1$ by the alternating series test, and so the power series representation is valid on $[-1, 1]$. Since we have found a power series representation of $f(x)$, this must be its Taylor series.

5. Polynomial Approximations

Determining when a function is equal to its Taylor series is quite a subtle question. In fact, it's also quite hard: in general, there is not much we can say. At the very minimum however, we can say the following. For each fixed value of x, consider the n-th partial sum $T_n(x)$ of the Taylor series $T(x)$, that is $T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k$. Saying that $f(x) = T(x)$ is the same as saying that $T_n(x) \to f(x)$ as $n \to \infty$. If we set $R_n(x) = f(x) - T_n(x)$, this says that if $f(x) = T(x)$, if and only if $R_n(x) \to 0$ as $n \to \infty$.

Theorem 5.1 (Representation by Taylor series)**.** *An infinitely differentiable function $f(x)$ can be written as a power series centered at c if and only if the n-th order remainder term $R_n(x) = f(x) - T_n(x)$ satisfies $\lim_{n\to\infty} R_n(x) = 0$ for all $x \in I$.*

To reiterate for emphasis, this theorem is saying very little: we merely translated the statement that $f(x) = T(x)$ into a statement about its partial sums via the definition of convergence of an infinite series. Explicitly computing the remainder term $R_n(x)$ is generally a hopeless task. Therefore if one wants to check using the above criterion that $f(x)$ can be represented by a power series, we need to come up with a way of estimating the remainder term $R_n(x)$ if we want to see if it tends to 0.

Definition 5.2. The n-th order **Taylor polynomial** of $f(x)$ centered at c is the n-th partial sum of its Taylor series, $T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k$.

Example 5.3. Let's compute the 4-th order Taylor polynomial of $f(x) = xe^{x^2}$ centered at $c = 0$. By definition, this is given by $f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}x^3 + \frac{f^{(4)}(0)}{24}x^4$. One such approach is to just calculate all the relevant derivatives and plug in $x = 0$. It's an easy computation to check that $f'(x) = (2x^2+1)e^{x^2}$, $f''(x) = (4x^3+6x)e^{x^2}$, $f'''(x) = (8x^4+24x^2+6)e^{x^2}$ and $f^{(4)}(x) = (16x^5 + 80x^3 + 60x)e^{x^2}$. Plugging in 0 then gives $T_4(x) = x + x^3$. Another way we could have done this computation is as follows. $T_4(x)$ is the 4-th degree polynomial that comes from the Taylor series of $f(x)$, so if we compute it's Taylor series, we can chop off terms to get the Taylor polynomial. Since $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, we have $xe^{x^2} = x\sum_{n=0}^{\infty} \frac{(x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{n!} = x + x^3 + \frac{x^5}{2} + \ldots$. We recover $T_4(x)$ by chopping off the sum at the degree 4 term (of which we see there is none), so $T_4(x) = x + x^3$.

This example illustrates several things: using Taylor series to compute Taylor polynomials is significantly faster, that the n-th degree Taylor polynomial doesn't even need to have degree n, and that two Taylor polynomials could be equal (here we have $T_3(x) = T_4(x)$).

Example 5.4. Let $f(x) = x^4 + \frac{1}{2}x^2 - 1$. Then the 4-th order Taylor polynomial of $f(x)$ centered at $c = 0$ is just $x^4 + \frac{1}{2}x^2 - 1$. This is because $f(x)$ is *already* a degree 4 polynomial.

The remainder term $R_n(x)$ tells you how far off from $f(x)$ the approximation $T_n(x)$ is. Determining how "good" of an approximation Taylor polynomials are is one of the major theorems of calculus!

Theorem 5.5 (Taylor's Theorem). *Let f be a function such that $f^{(n+1)}(x)$ exists and is continuous. Suppose there is a constant K_{n+1} such that $|f^{(n+1)}(z)| \leq K_{n+1}$ for all z between x and c. Then $|R_n(x)| = |f(x) - T_n(x)| \leq \frac{K_{n+1}}{(n+1)!}|x-c|^{n+1}$.*

Taylor's inequality tells us that the size of the remainder term $R_n(x)$ depends on the size of the $(n+1)$-st derivative of f. This makes it more explicit *why* it's hard to show a function can be written as a power series: computing arbitrary order derivatives is generally not easy.

Example 5.6. Consider the Taylor expansion of e^x centered at $c = 0$. Since $\frac{d^n}{dx^n}e^x = e^x$ for all $n \geq 0$, Taylor's theorem says that for any value of x and any $n \geq 0$, we have $|R_n(x)| \leq \frac{e^{|x|}}{(n+1)!}|x|^{n+1}$ because the maximum value of e^x on the interval $[-x, x]$ happens at whichever endpoint is positive. Taking a limit as $n \to \infty$ then shows that $R_n(x) \to 0$, and so this means the Taylor series of e^x converges to e^x, so we get an equality $e^x = \sum_{n=0}^{\infty} \frac{1}{n!}x^n$. Some calculus books give this as the *definition* of the exponential function. In particular, we get the numerical identity $e = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \ldots$.

As a first application, we can handle one simple case of when a function can be written as a power series: let $T(x)$ be the Taylor series of $f(x)$ centered at c, and suppose that $T(x)$ has radius of convergence R. If there is some number K such that $|f^{(n)}(x)| \leq K$ for all n and all x such that $|x - c| < R$, then applying Taylor's inequality says that $|R_n(x)| \leq \frac{K}{(n+1)!}|x-c|^{n+1} \leq \frac{K}{(n+1)!}R^{n+1}$. Taking $n \to \infty$ says that $|R_n(x)| \to 0$, so that we have proved the following:

Theorem 5.7. *Let f be an infinitely differentiable function. Let $T(x)$ be the Taylor series of $f(x)$ centered at c with radius of convergence R. Suppose there is some constant K such that $|f^{(n)}(x)| \leq K$ for all n and all x such that $|x - c| < R$. Then $f(x) = T(x)$ for $|x - c| < R$. That is to say, such an f has a power series representation.*

Example 5.8. With $f(x) = \sin(x)$, we see that $|f^{(n)}(x)| \leq 1$ for all $n \geq 0$. The above theorem says that the Taylor series of $\sin(x)$ converges to $\sin(x)$, and so from our prior example we have an actual equality $\sin(x) = \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{(2k+1)!}$, valid for all x because the series has infinite radius of convergence. We can compute the Taylor series of $\cos(x)$ by taking the derivative of the Taylor series of $\sin(x)$. Doing so, we find $\cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}x^{2k}$, which is again valid for all x, so we have the equality $\cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}x^{2k}$.

We now give some examples of how the error bound inequality can be used to quantify how "good" a polynomial approximation is.

Example 5.9. Suppose $f(x) = e^{-x}$ and we have $T_3(x)$ centered at $c = 1$. How good of an approximation to $f(1.1)$ is $T_3(1.1)$? The error bound formula says $|f(1.1) - T_3(1.1)| \leq \frac{K_4}{4!}|.1|^4$, where K_4 is an upper bound of $|f^{(4)}(x)|$ on the interval $[1, 1.1]$. Since $|f^{(4)}(x)| = f^{(4)}(x) = e^{-x}$, K_4 is just an upper bound of e^{-x} on the interval $[1, 1.1]$. The function e^{-x} is strictly decreasing on this interval, so it attains it's maximal value on the interval at the left endpoint $x = 1$. This says the maximal value is given by $\frac{1}{e}$, so we can take $K_4 = \frac{1}{e}$.

Although this is a perfectly valid choice of K_4, if the point is to do an approximation *by hand*, it's completely useless to choose a value of K_4 that would involve e, since that's another thing we have to approximate. Since $e \approx 2.718$, in particular we have $e \geq \frac{5}{2}$ so $\frac{1}{e} \leq \frac{2}{5}$. We will then instead take $K_4 = \frac{2}{5}$, the trade off being the error estimate will be a little bit worse, but computable by hand. Plugging into the error bound formula, this says $|f(1.1) - T_3(1.1)| \leq \frac{2/5}{24}(.1)^4 = \frac{1}{60 \cdot 10^4} \approx .000001$. This says if we want to estimate $e^{-1.1}$, that $T_3(1.1)$ is a very good estimate. Indeed, we can compute that $T_3(x)$ centered at $c = 1$ is given by $T_3(x) = \frac{1}{e} - \frac{1}{e}(x-1) + \frac{1}{2e}(x-1)^2 - \frac{1}{6e}(x-1)^3$, so that $T_3(1.1) = \frac{1}{e} - \frac{1}{10e} + \frac{1}{200e} - \frac{1}{6000e} = \frac{5429}{6000e}$. If we use the approximation $e \approx 2.718$, then $T_3(1.1) \approx .3329$, while $f(1.1) \approx .33287$ (we obviously lost a bit more precision by having to approximate e).

Example 5.10. Suppose $f(x) = xe^{x^2}$. Earlier, we computed that $T_3(x) = x + x^3$. How good of an approximation is this to $f(x)$? For an arbitrary value $x > 0$, the error bound formula says $|f(x) - T_3(x)| \leq \frac{K_4}{4!}x^4$, where K_4 is an upper bound of $|f^{(4)}(z)|$ on the interval $[0, x]$. We also computed that $g(z) = |f^{(4)}(z)| = f^{(4)}(z) = (16z^5 + 80z^3 + 60z)e^{z^2}$. By definition, K_4 is an upper bound of this function on the interval $[0, x]$. The function $g(z)$ is strictly increasing, because $g'(z) = (32z^6 + 240z^4 + 360z^2 + 60)e^{z^2} \geq 0$ when z is in the interval $[0, x]$. In particular, this says $g(z)$ attains it's maximal value at the right endpoint of this interval, i.e. at $z = x$. Therefore, we may choose $K_4 = (16x^5 + 80x^3 + 60x)e^{x^2}$. If we plug this in, this says $|f(x) - T_3(x)| \leq \frac{(16x^5+80x^3+60x)e^{x^2}}{24}x^4 = \frac{(16x^9+80x^7+60x^5)e^{x^2}}{24}$.

This says at worst, the error grows at the same rate as the function $\frac{(16x^9+80x^7+60x^5)e^{x^2}}{24}$. Since $\lim_{x \to 0} \frac{(16x^9+80x^7+60x^5)e^{x^2}}{24} = 0$ (and it goes to 0 quite quickly), for values of x close to 0, the approximation will be quite good. For example, using a calculator we find $|f(.1) - T_3(.1)| \leq .00002559$, so $T_3(.1) = .101$ approximates $f(.1)$ to within 4 decimal places. Indeed, we see $f(.1) \approx .101005$. However, as $x \to \infty$, we have $\frac{(16x^9+80x^7+60x^5)e^{x^2}}{24} = \infty$, and moreover, this function is growing extremely quickly (faster than an exponential function!). This says for values of x far from 0, the error bound will be terrible. For example, at $x = 1$, we see $|f(1) - T_3(1)| \leq \frac{13e}{2} \approx 17.668$. This an absolutely useless estimate, because we knew $f(1) = e$ and $T_3(1) = 2$, so in actuality $|f(1) - T_3(1)| \approx .718$!

Example 5.11. Suppose we want to compute $\ln(1.1)$ to within 4 decimal places of accuracy. How can we do this? One such approach using what we have done so far is to figure out how many terms in the Taylor series of $\ln(1.1)$ are necessary in order for the N-th Taylor polynomial $T_N(x)$ to approximate to within that level of error by using the error bound formula. In order for the error bound formula to remain useful, we must make sure that we center $T_N(x)$ somewhere close to 1.1. One such approach is to center it at $c = 1$, so that phrased mathematically, we want to find N such that $|f(1.1) - T_N(1.1)| \leq \frac{1}{10^4}$, where $f(x) = \ln(x)$ and $T_N(x)$ is centered at $c = 1$.

The error bound formula says $|f(1.1) - T_N(1.1)| \leq \frac{K_{N+1}}{(N+1)!}(.1)^{N+1}$, so if we make this smaller than $\frac{1}{10^4}$, then we're good. In order to do this, we need to figure out how to pick K_{N+1}. By definition, K_{N+1} is an upper bound of $|f^{(N+1)}(x)|$ on the interval $[1, 1.1]$. First, we compute an arbitrary order derivative of $f(x)$. We have $f(x) = \ln(x)$, $f'(x) = x^{-1}$, $f''(x) = -x^{-2}$, $f'''(x) = 2x^{-3}$, $f^{(4)}(x) = -6x^{-4}$, and so on. Continuing the pattern, we see that $f^{(N)}(x) = (-1)^{N+1}(N-1)!x^{-N}$, so that $|f^{(N+1)}(x)| = N!x^{-(N+1)}$. In particular, this function is decreasing (because the derivative is always negative), so that it's maximal value happens at the left endpoint $x = 1$. Plugging this in, the maximum value of $N!x^{-(N+1)}$ is just $N!$, so we may take $K_{N+1} = N!$. We then need to solve the inequality $\frac{K_{N+1}}{(N+1)!}\frac{1}{10^{N+1}} \leq \frac{1}{10^4}$. Plugging in our choice of K_{N+1}, this is the same thing as solving $\frac{1}{(N+1)10^{N+1}} \leq \frac{1}{10^4}$, i.e. $10^4 \leq (N+1)10^{N+1}$. We see that $N = 3$ is the smallest such choice of N that works, so we only need to take 3 terms in the Taylor series to get the desired level of accuracy.

If we wanted to then approximate $\ln(1.1)$, we can then easily compute $T_3(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3$, so that $T_3(1.1) = \frac{1}{10} - \frac{1}{200} + \frac{1}{3000} = \frac{143}{1500} \approx .09533$, and indeed, this is a good approximation to $\ln(1.1) \approx .09531$.

6. Applications of Taylor Series

Taylor series are the ultimate tool of calculus – they can be used to answer almost all classical calculus problems you might be interested in solving. In particular, we will see how Taylor series can be used to do the following:

- Compute limits.
- Compute derivatives at a point.
- Compute the value of an infinite series.
- Approximate the value of a definite integral when the integrand does not have an anti-derivative we can write down.
- Analyze the growth rate of functions, making it easier to apply the limit comparison test.

Example 6.1. Suppose we want to compute $\lim_{x\to 0} \frac{\sin(x^4)-x^4}{(x^4-\frac{1}{6}x^3)^4}$. If you try and use L'Hopital's rule, you'll very quickly convince yourself that it will be extremely difficult. How else can we compute this limit? One approach is replace the numerator with it's Taylor series, and do the resulting limit computation. We have $\sin(x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}$, so $\sin(x^4) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{8n+4}}{(2n+1)!} = x^4 - \frac{1}{6}x^{12} + \ldots$, thus $\sin(x^4) - x^4 = \sum_{n=1}^{\infty}(-1)^n \frac{x^{8n+4}}{(2n+1)!} = -\frac{1}{6}x^{12} + \frac{1}{120}x^{20} + \ldots$, so that $\lim_{x\to 0} \frac{\sin(x^4)-x^4}{(x-\frac{1}{6}x^3)^4} = \lim_{x\to 0} \frac{-\frac{1}{6}x^{12}+\frac{1}{120}x^{20}+\ldots}{(x^4-\frac{1}{6}x^3)^4}$. Dividing the numerator and denominator through by x^{12}, we find $\lim_{x\to 0} \frac{-\frac{1}{6}x^{12}+\frac{1}{120}x^{20}+\ldots}{(x^4-\frac{1}{6}x^3)^4} = \lim_{x\to 0} \frac{-\frac{1}{6}+\frac{1}{120}x^{8}+\ldots}{(x-\frac{1}{6})^4} = -6^3 = -216$.

Example 6.2. Let $f(x) = e^{x^2}$. Suppose we wanted to calculate the 1000-th derivative of f at 0, $f^{(1000)}(0)$. It's obviously impossible to calculate 1000 derivatives by hand, and no computer will be able to calculate the derivative explicitly. How can we do this? The easiest way is to compute the Taylor series of $f(x)$ centered at 0, which encodes information about all derivatives of f at 0. We have $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, so that $e^{x^2} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$. In particular, the definition of the Taylor series says $e^{x^2} = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$, so to recover the value of $f^{(1000)}(0)$, we need to look at the coefficient of x^{1000} in the Taylor series of $f(x)$. We see that the coefficient is just $\frac{1}{500!}$, so by comparing coefficients in these two series we find $\frac{1}{500!} = \frac{f^{(1000)}(0)}{1000!}$, which says $f^{(1000)}(0) = \frac{1000!}{500!}$.

Example 6.3. The series $\sum_{n=0}^{\infty} \frac{2n+1}{4^n}$ converges by doing a direct comparison test with a geometric series. As it turns out, we can actually compute the value of this sum. The way we do this is as follows: write $\sum_{n=0}^{\infty} \frac{2n+1}{4^n} = \sum_{n=0}^{\infty}(2n+1)(\frac{1}{4})^n$. Then the value of this sum is $f(1/4)$, where $f(x) = \sum_{n=0}^{\infty}(2n+1)x^n$. If we can find a function whose Taylor series centered at 0 is equal to $f(x)$, then we can find the exact value of the series. It's a quick check to see that $f(x)$ converges for $|x| < 1$, so we can write $f(x) = 2\sum_{n=0}^{\infty} nx^n + \sum_{n=0}^{\infty} x^n$ for such values. We know that the second series is the Taylor series of $\frac{1}{1-x}$. What about the first series? The key observation is that this series is "almost" what you would get if you differentiated the second series. In fact, the only difference is a factor of x, so we have $\sum_{n=0}^{\infty} nx^n = x\frac{d}{dx}\sum_{n=0}^{\infty} x^n = x\frac{d}{dx}\frac{1}{1-x} = \frac{x}{(1-x)^2}$. This tells us that $f(x) = \frac{2x}{(1-x)^2} + \frac{1}{1-x}$, for $|x| < 1$. Plugging in $x = \frac{1}{4}$ then says $f(1/4) = \sum_{n=0}^{\infty} \frac{2n+1}{4^n} = \frac{20}{9}$.

Example 6.4. At some point in your life, someone has probably mentioned that the function $f(x) = e^{-x^2}$ does not have an anti-derivative that you can write down. However, functions that look like this are of fundamental importance in fields relating to mathematics, for example, $f(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$ is what is known as the "standard normal distribution" in statistics. If you have taken such a course before, then you know that computing definite integrals of this function is extremely important. How can we do it? As an example, we'll approximate $\int_{-1}^{1} \frac{1}{\sqrt{2\pi}}e^{-x^2/2}\,dx$. A statistical interpretation of this integral is that if you have a population that is normally distributed, this integral computes the proportion of the population that lies within 1 standard deviation from the mean.

First, we'll find a Taylor series of the integrand centered at 0. This is easy to do: we know $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, so $e^{-x^2/2} = \sum_{n=0}^{\infty} \frac{(-x^2/2)^n}{n!} = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{2^n n!}$. Integrating this series then says $\int_{-1}^{1} \frac{1}{\sqrt{2\pi}}e^{-x^2/2}\,dx = \frac{1}{\sqrt{2\pi}}\int_{-1}^{1}\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{2^n n!}\,dx = \frac{1}{\sqrt{2\pi}}\sum_{n=0}^{\infty}\int_{-1}^{1}(-1)^n\frac{x^{2n}}{2^n n!}\,dx = \frac{1}{\sqrt{2\pi}}\sum_{n=0}^{\infty}(-1)^n\frac{2}{(2n+1)2^n n!}$. In order to approximate the sum, we can use the remainder estimate from the alternating series test. First, let's pick a desired level of accuracy, say, 4 decimal places. If we let S denote the value of the series, then the alternating series test says $|S - S_N| \leq a_{N+1}$. Then we would like to find N such that $a_{N+1} \leq \frac{1}{10^4}$, that is, solve $\frac{2}{(2N+3)2^{N+1}(N+1)!} \leq \frac{1}{10^4}$, which is equivalent to solving $20000 \leq (2N+3)2^{N+1}(N+1)!$. We see that $N = 4$ is the smallest value of N that works, so we arrive at the desired accuracy by computing S_4. Using a calculator to do so, we then conclude that $\int_{-1}^{1} \frac{1}{\sqrt{2\pi}}e^{-x^2/2}\,dx \approx .6827$.

Example 6.5. Suppose we wanted to know if $\sum_{n=1}^{\infty}(1/n - \tan^{-1}(1/n))$ converges or diverges. In order to come to a conclusion, we need to understand how the summand grows as $n \to \infty$. The 3rd order Taylor polynomial of $\tan^{-1}(x)$ centered at $x = 0$ is $\tan^{-1}(x) \approx x - \frac{1}{3}x^3$. Since $1/n \to 0$ as $n \to \infty$, this means that $\tan^{-1}(1/n) \approx \frac{1}{n} + \frac{1}{3n^3}$, so as $n \to \infty$ we see that $1/n - \tan^{-1}(1/n) \approx \frac{1}{3n^3}$. Therefore, our original sum should converge.

To formally show this, we'll examine $\sum_{n=1}^{\infty} |1/n - \tan^{-1}(1/n)|$ instead (because showing the terms in our original sum are *positive* so we can use a comparison test is rather tricky!). By Taylor's theorem, we have $\tan^{-1}(x) = x - \frac{1}{3}x^3 + R_3(x)$, where $|R_3(x)| \leq Cx^4$ for some constant C. Therefore, $\tan^{-1}(1/n) = \frac{1}{n} - \frac{1}{3n^3} + R_3(1/n)$, with $|R_3(1/n)| \leq \frac{C}{n^4}$. We have $\lim_{n\to\infty}\frac{|1/n - \tan^{-1}(1/n)|}{1/(3n^3)} = \lim_{n\to\infty}\frac{|1/(6n^3) - R_3(1/n)|}{1/(6n^3)} \leq 1 + \lim_{n\to\infty}\frac{|R_3(1/n)|}{1/(6n^3)} \leq 1 + \lim_{n\to\infty}\frac{6C}{n} = 1$. Similarly, we also see that $\lim_{n\to\infty}\frac{|1/n - \tan^{-1}(1/n)|}{1/(3n^3)} \geq \lim_{n\to\infty}\frac{1/n - \tan^{-1}(1/n)}{1/(3n^3)} = 1$, and so we conclude that $\lim_{n\to\infty} \frac{|1/n-\tan^{-1}(1/n)|}{1/(3n^3)} = 1$. Therefore by the limit comparsion test, $\sum_{n=1}^{\infty} |1/n - \tan^{-1}(1/n)|$ converges, and so the original series does too.