Sequences

A <u>sequence</u> is a function

$f: \mathbb{N} \to \mathbb{R}$

We usually write a_n for $f(n)$ and think of the list of all values $\{a_n\}$ as the sequence.

<u>Ex</u>: • $a_n = 1/n^2$ $\quad 1, 1/4, 1/9, 1/16, 1/25, \ldots$

• $F_n = F_{n-1} + F_{n-2}$ $\quad F_1, F_2 = 1$ $\quad 1, 1, 2, 3, 5, 8, 13, \ldots$ Fibonacci Sequence

Since sequences are functions, we can do all the usual operations: add/mult/divide, take limits, etc.

Goal: do discrete calculus

Continuous	Discrete
functions	Sequences
$\frac{d}{dx}$	Δ
$\int$	$\sum$

$\Delta a_n = a_{n+1} - a_n$ discrete derivative.

$\sum_{n=a}^{b} a_n$ discrete integral.

there are analogues of the techniques we've learned for the discrete world:

L'Hopital $\longleftrightarrow$ Stoltz - Cesaro

IBP $\longleftrightarrow$ Summation by parts but beyond what we're going to talk about.

Some sequences that don't have easy analogues in the continuous realm:

$(-1)^n$ No analogue!

$n! = n\cdot(n-1)\cdot\dots\cdot 1 \quad \rightsquigarrow \quad \Gamma(x)$ (HW 4)

Factorials will come up very often.

Growth rates of common sequences:

$$\ln(n) \ll \underset{a>0}{n^a} \ll \underset{b>1}{b^n} \ll n! \ll n^n$$

Analyzing sequences w/ $(-1)^n$ usually requires the Squeeze thm.

Note: if $a_n = f(n)$ comes from a cts function on $\mathbb{R}$, can still use L'Hopital to compute limits, e.g. $\lim_{n\to\infty} \frac{n^3+n^2+1}{e^n}$

Useful limit to know: for any R,

$$\lim_{n\to\infty} \frac{R^n}{n!} = 0.$$

Ex: $a_n = \frac{(-1)^n n + 2^n}{n!}$ what is $\lim_{n\to\infty} a_n$?

Sep into pos/neg parts:

$$\frac{-n+2^n}{n!} \leq a_n \leq \frac{n+2^n}{n!}$$

$\lim_{n\to\infty} \frac{n+2^n}{n!}$

$= \lim_{n\to\infty} \frac{1}{(n-1)!} + \frac{2^n}{n!} = 0$

Also $\to 0$

So $\lim_{n\to\infty} a_n = 0$ by squeeze thm.

Def: $\{a_n\}$ is called bounded above if $a_n \le M$ for some M. Similarly, bounded below if $a_n \ge m$ for some m and bounded if bounded above + below.

Def: $\{a_n\}$ is called monotonically inc. if $a_n \le a_{n+1}$ for all n

monotonically dec if $a_n \ge a_{n+1}$ for all n

monotonic if one of these holds.

Thm: bounded, monotonic sequence converges. That is, if $\{a_n\}$ is bounded, mono. then $\lim_{n\to\infty} a_n$ exists.

This thm is very useful for analyzing sequences that are recursively defined.

Ex: $a_1 = 2$
$a_{n+1} = \frac{1}{2}(a_n + 6)$ for $n = 1, 2, \ldots$

Fact: $\{a_n\}$ is bounded above by 6. Informally, if $a_n \leq 6$ then $a_{n+1} = \frac{1}{2}(a_n+6) \leq \frac{1}{2}(6+6) = 6$

So since $a_1 < 6$ all terms are (but need induction to formally show this).

Note $a_n \leq a_{n+1} \iff a_n \leq \frac{1}{2}(a_n+6) \iff \frac{1}{2}a_n \leq 3 \iff a_n \leq 6$

So $\{a_n\}$ being bounded is equivalent to it being mono. inc. here

by thm, $\lim_{n\to\infty} a_n = L$ exists. The recursive def says

$$L = \frac{1}{2}(L+6) \text{ so } L = 6 \text{ and } \lim_{n\to\infty} a_n = 6.$$

Key point: we didn't need an explicit formula to do this!