Improper Integrals II

Useful growth rates: $f(x) << g(x)$ if eventually $f(x) \le g(x)$, i.e. there is x_0 s.t. $x \geq x_0$ means $f(x) \le g(x)$.

$$\ln(x) << x^a << b^x$$
$$a>0 \qquad b>1$$

functions like sin/cos/tan^{-1} are bounded, and so generally will not impact convergence or not.

How can we check if an integral converges or not when we can't explicitly integrate?

<u>Comparison test</u>: • if $\int f(x)\,dx \le \int g(x)\,dx$ and $\int g(x)\,dx$ converges, then $\int f(x)\,dx$ converges.

- If $\int g(x)\,dx \le \int f(x)\,dx$ and $\int g(x)\,dx$ diverges, then $\int f(x)\,dx$ diverges.

A useful class of functions to compare w/ are $\frac{1}{x^p}$ for $p>0$:

$(a>0)$

Ex: $\int_a^\infty \frac{1}{x^p}\,dx = \frac{x^{1-p}}{1-p}\Big|_a^\infty$

$$\lim_{R\to\infty} \frac{R^{1-p}}{1-p} - \frac{1}{1-p} = \begin{cases} \frac{a^{1-p}}{1-p} & p>1 \\ \infty & 0<p\le 1 \end{cases}$$

Similarly, $\int_0^a \frac{1}{x^p}\,dx = \begin{cases} \infty & p>1 \\ -\infty & p=1 \\ \frac{a^{1-p}}{1-p} & 0<p<1 \end{cases}$

Ex: Does $\int_1^{\infty} \frac{1}{x^3+x^2+1}\,dx$ conv. or div?

As $x \to \infty$, $x^3+x^2+1 \approx x^3$ and $\int_1^{\infty} \frac{1}{x^3}\,dx$ conv. So it should conv.

To show this, need an inequality:

$x^3+x^2+1 \geq x^3$ for $x \geq 1$ clearly, so

$$\frac{1}{x^3+x^2+1} \leq \frac{1}{x^3} \Rightarrow \int_1^{\infty} \frac{1}{x^3+x^2+1}\,dx \leq \int_1^{\infty} \frac{1}{x^3}\,dx < \infty.$$

So by comparison test, $\int_1^{\infty} \frac{1}{x^3+x^2+1}\,dx$ Converges.

Ex: $\int_0^1 \frac{e^{-x}}{\sqrt{x^5+x^7}}\,dx$ blows up at $x=0$.

Near $x=0$, $x^5+x^7 \simeq x^5$, and $e^{-x} \simeq 1$, so looks like

$$\int_0^1 \frac{1}{x^{5/2}}\,dx$$ which diverges, so

our int. should diverge. Need an inequality to prove it:

On $[0,1]$, $x^5+x^7 \leq 2x^5$

So $\sqrt{x^5+x^7} \leq \sqrt{2}\, x^{5/2}$.

Also, $e^{-x} \geq 1/e$, so

$$\frac{e^{-x}}{\sqrt{x^5+x^7}} \geq \frac{1}{e} \cdot \frac{1}{\sqrt{x^5+x^7}} \geq \frac{1}{e\sqrt{2}} \cdot \frac{1}{x^{5/2}}$$

$$\int_0^1 \frac{e^{-x}}{\sqrt{x^5+x^7}}\,dx \geq \frac{1}{e\sqrt{2}}\int_0^1 \frac{1}{x^{5/2}}\,dx = \infty$$

So it diverges by comparison test.

Ex: $\displaystyle\int_1^\infty \frac{1}{x+e^x}\,dx$

As $x \longrightarrow \infty$, $\dfrac{1}{x+e^x} \approx \dfrac{1}{e^x}$

and $\displaystyle\int_1^\infty \frac{1}{e^x}\,dx$ converges by

Computing the integral.

Now, $x+e^x \geq e^x$ so

$$\frac{1}{x+e^x} \le \frac{1}{e^x} \quad \text{so that}$$

$$\int_1^\infty \frac{1}{x+e^x}\,dx \le \int_1^\infty \frac{1}{e^x}\,dx < \infty$$

which means it converges by

comparison test.