Improper Integrals I

The Riemann integral cannot integrate over "infinite" regions:

- Infinite Regions. (I)

- Functions that "blow up" on the interval (II)

Goal: extend the integral to handle these cases.

Type I: Infinite Region.

$(-\infty, a]$, $[a, \infty)$ and $(-\infty, \infty)$.

Start with $[a, \infty)$:

Integrate over larger and larger finite intervals and define as a limiting process.

$$\int_a^\infty f(x)\,dx \stackrel{\text{def}}{=} \lim_{R\to\infty} \int_a^R f(x)\,dx$$

$$\int_{-\infty}^a f(x)\,dx \stackrel{\text{def}}{=} \lim_{R\to\infty} \int_R^a f(x)\,dx$$

$$\int_{-\infty}^\infty f(x)\,dx \stackrel{\text{def}}{=} \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx$$

for any $c \in (-\infty, \infty)$.

$$\lim_{R_1 \to -\infty} \int_{R_1}^{c} f(x)\,dx + \lim_{R_2 \to \infty} \int_{c}^{R_2} f(x)\,dx$$

<u>Note</u>: this is <u>not</u> $\lim_{R \to \infty} \int_{-R}^{R} f(x)\,dx$.

This "symmetric" process is called the Cauchy Principal value.

<u>Type II</u>: Discontinuity.

Very similar idea. Suppose that $f(x)$ has an infinite discontinuity at c

$c \in (a, b)$.

$$\int_a^b f(x)\,dx = \lim_{R_1 \to c} \int_a^{R_1} f(x)\,dx + \lim_{R_2 \to c} \int_{R_2}^{b} f(x)\,dx.$$

Discontinuity at endpoint is handled similarly to above.

If the limit is defined, we say the improper integral <u>converges</u> and assign it the value of the limit. Otherwise, it <u>diverges</u>.

<u>Ex:</u> $\int_0^{\infty} e^{-x}\,dx$

$$= \lim_{R\to\infty} \int_0^R e^{-x}\,dx \overset{\lim_{R\to\infty}}{=} -e^{-x}\Big]_0^R$$

$$= \lim_{R\to\infty} 1 - e^{-R} = 1.$$

Ex: $$\int_{-\infty}^{\infty} \frac{1}{\pi(1+x^2)}\,dx = \lim_{R_1\to-\infty} \int_{R_1}^{0} \frac{1}{\pi(1+x^2)}\,dx$$

$$+ \lim_{R_2\to\infty} \int_0^{R_2} \frac{1}{\pi(1+x^2)}\,dx$$

$$= \lim_{R_1\to-\infty} \frac{1}{\pi}\tan^{-1}(x)\Big|_{R_1}^{0} + \lim_{R_2\to\infty} \frac{1}{\pi}\tan^{-1}(x)\Big|_0^{R_2}$$

$$= \frac{1}{2} + \frac{1}{2} = 1.$$

Ex: $\displaystyle\int_0^2 \frac{1}{(x-1)^{2/3}}\,dx = \int_0^1 \frac{1}{(x-1)^{2/3}}\,dx + \int_1^2 \frac{1}{(x-1)^{2/3}}\,dx$

$$= \lim_{R_1 \to 1} \int_0^{R_1} \frac{1}{(x-1)^{2/3}}\,dx + \lim_{R_2 \to 1} \int_{R_2}^{2} \frac{1}{(x-1)^{2/3}}\,dx$$

$$= \lim_{R_1 \to 1} 3(x-1)^{1/3}\Big|_0^{R_1} + \lim_{R_2 \to 1} 3(x-1)^{1/3}\Big|_{R_2}^{2}$$

$$= \lim_{R_1 \to 1} \left[3(R_1-1)^{1/3} + 3\right] + \lim_{R_2 \to 1} \left[3 - 3(R_2-1)^{1/3}\right]$$

$$= 6.$$