Numerical Integration

Most functions cannot be integrated explicitly.

e.g. $\int e^{-\frac{1}{2}x^2} dx$ has no closed form antiderivative.

But we often need to compute definite integrals of these expressions.

Two questions:

1. How to approximate?
2. How good is the approximation?

Basic ideas: Choose sample points and fit polynomials through them.

- Midpoint Rule — degree 0
- Trapezoidal Rule — degree 1
- Simpson's Rule — degree 2

Midpoint:

$$\int_a^b f(x)\,dx \approx \Delta x\left(f(x_1) + \dots + f(x_N)\right)$$

$$X_i = a + \left(i - \tfrac{1}{2}\right) \cdot \Delta x \qquad \Delta X = \frac{b-a}{N}$$

Trapezoidal:

$$\int_a^b f(x)\,dx \approx \frac{1}{2}\Delta x\left(y_0 + 2y_1 + \dots + 2y_{N-1} + y_N\right)$$

$$y_i = f(x_i) \qquad x_i = a + i\Delta x$$

Simpson's:

$$\int_a^b f(x)\,dx \approx \frac{1}{3}\Delta x\left[y_0 + 4y_1 + 2y_2 + \dots + 2y_{N-2} + 4y_{N-1} + y_N\right]$$

$$\Delta x = \frac{b-a}{N} \qquad y_i = f(x_i) \qquad x_i = a + i\Delta x$$

Note: need <u>even</u> N for Simpson's rule to work.

<u>Ex:</u> Approx. $\int_2^4 \sqrt{1+x^3}\,dx$ using Midpoint, Trap., Simpson's for $N=6$.

$\Delta x = \frac{4-2}{6} = \frac{1}{3}$

Points: $2, \frac{7}{3}, \frac{8}{3}, 3, \frac{10}{3}, \frac{11}{3}, 4$

<u>Midpoint:</u>

$\frac{13}{6}, \frac{15}{6}, \frac{17}{6}, \frac{19}{6}, \frac{21}{6}, \frac{23}{6}$

$$\frac{1}{3}\cdot\left[f\left(\frac{13}{6}\right)+f\left(\frac{15}{6}\right)+f\left(\frac{17}{6}\right)+f\left(\frac{19}{6}\right)+f\left(\frac{21}{6}\right)+f\left(\frac{23}{6}\right)\right]$$

$$\approx 10.73707$$

Trap:

$$\frac{1}{6}\cdot\left[f(2)+2f\left(\tfrac{7}{3}\right)+2f\left(\tfrac{8}{3}\right)+2f(3)+\right.$$
$$\left.2f\left(\tfrac{10}{3}\right)+2f\left(\tfrac{11}{3}\right)+f(4)\right]$$

$$\approx 10.75063$$

Simpson:

$$\frac{1}{9}\cdot\left[f(2)+4f\left(\tfrac{7}{3}\right)+2f\left(\tfrac{8}{3}\right)+4f(3)\right.$$
$$\left.+2f\left(\tfrac{10}{3}\right)+4f\left(\tfrac{11}{3}\right)+f(4)\right]$$

$$\approx 10.74159$$

which is going to be best?

Error Analysis:

Midpoint:

$$\text{Error}(M_N) \leq \frac{K_2}{24} \frac{(b-a)^3}{N^2}$$

K_2 upper bound of

$|f''(x)|$ on $[a,b]$.

$$\text{Error}(T_N) \le \frac{K_2}{12}\frac{(b-a)^3}{N^2}$$

$$\text{Error}(S_N) \le \frac{K_4}{180N^4}(b-a)^5$$

Simpson's Rule is generally very accurate.

Ex: Find N s.t. S_N approx.

$\int$

$$\int_1^{5} \ln(x)\,dx \text{ to within } 10^{-6}?$$

Need N s.t.

$$\frac{K_4}{180N^4}\cdot 4^5 \le \frac{1}{10^6}$$

$$|f^{(4)}(x)| = \frac{6}{x^4}$$

Max value of 6 on $[1,5]$

$$\frac{6}{180N^4}\cdot 4^5 \le \frac{1}{10^6}$$

$N \approx 77.$

Need N even so

$$\boxed{N = 78}$$