Partial Fractions

- Algorithm for integrating rational functions, i.e. $\frac{p(x)}{q(x)}$ for $p(x), q(x)$ polynomials.

- If degree $p(x) >$ degree $q(x)$, do long division first

- $q(x)$ factors into product of linear factors and irred. quadratic factors

$(x-a)^N$ contributes $\frac{A_1}{x-a} + \dots + \frac{A_N}{(x-a)^N}$

$(x^2+ax+b)^N$ contributes $\frac{A_1x+B_1}{(x^2+ax+b)} + \dots + \frac{A_Nx+B_N}{(x^2+ax+b)^N}$

- Algebraically find decomp. once the "shape" is known.

Ex: $$\frac{x^2+4x+12}{(x+2)(x^2+4)} = \frac{A}{x+2} + \frac{Bx+C}{x^2+4}$$

$$x^2+4x+12 = A(x^2+4) + (x+2)(Bx+C)$$

$$= (A+B)x^2 + 2Bx + (4A+2C)$$

$$\begin{cases} A+B=1 \\ 2B+C=4 \\ 4A+2C=12 \end{cases} \Rightarrow \begin{matrix} A+B=1 \\ A-B=1 \end{matrix} \Rightarrow \begin{matrix} A=1 \\ B=0 \\ C=4 \end{matrix}$$

$$\frac{x^2+4x+12}{(x+2)(x^2+4)} = \frac{1}{x+2} + \frac{4}{x^2+4}$$

<u>Ex:</u>

$$\int \frac{x^2-4x+8}{(x-1)^2(x-2)^2}\,dx$$

$$\frac{x^2-4x+8}{(x-1)^2(x-2)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x-2} + \frac{D}{(x-2)^2}$$

$$x^2-4x+8 = A(x-1)(x-2)^2 + B(x-2)^2 + C(x-2)(x-1)^2 + D(x-1)^2$$

Plug in $x=1$:

$\underline{5 = B}$

Plug in $x=2$:

$\underline{4 = D}$

Plug in $x=0$:

$8 = -4A + 20 - 2C + 4$

$2A + C = 8$

Plug in $x=3$:

$5 = 2A + 5 + 4C + 16$

$-8 = A + 2C$

$\underline{A = 8}$ $\underline{C = -8}$

$$\frac{x^2 - 4x + 8}{(x-1)^2(x-2)^2} = \frac{8}{x-1} + \frac{5}{(x-1)^2} - \frac{8}{x-2} + \frac{4}{(x-2)^2}$$

$$\Rightarrow \int \frac{x^2 - 4x + 8}{(x-1)^2(x-2)^2}\,dx = 8\ln|x-1| - \frac{5}{x-1} - 8\ln|x-2| - \frac{4}{x-2} + C$$

Ex: $\displaystyle\int \frac{x^3+2x^2+1}{x+2}\,dx$ Do long division first

$$\begin{array}{r} x^2 \\ (x+2)\overline{\smash{)}\,x^3+2x^2+1} \\ -\underline{\left(x^3+2x^2\right)} \\ 1 \end{array}$$

$$\int \frac{x^3+2x^2+1}{x+2}\,dx = \int x^2 + \frac{1}{x+2}\,dx$$

$$= \frac{1}{3}x^3 + \ln|x+2| + C.$$

Ex: $\displaystyle\int \frac{x^3+x+1}{(x^2+1)(x^2+2)}\,dx$

$$\frac{x^3+x+1}{(x^2+1)(x^2+2)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+2}$$

$$x^3+x+1 = (Ax+B)(x^2+2) + (Cx+D)(x^2+1)$$

$$= (A+C)x^3 + (B+D)x^2 + (2A+C)x + (2B+D)$$

$$\begin{cases} A+C=1 \\ B+D=0 \\ 2A+C=1 \\ 2B+D=1 \end{cases}$$

$D=-B$

$B=1$

$A=0$

$C=1$

$A=0$
$B=1$
$C=1$
$D=-1$

$$\frac{x^3+x+1}{(x^2+1)(x^2+2)} = \frac{1}{x^2+1} + \frac{x-1}{x^2+2}$$

$$\int \frac{1}{x^2+1} + \frac{x}{x^2+2} - \frac{1}{x^2+2}\,dx$$

$$= \tan^{-1}(x) + \frac{1}{2}\ln|x^2+2| - \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + C$$

If you have repeated irred. quadratic factor, often times have to combine w/ trig sub to compute integral.

Ex:

$$\int \frac{4-x}{x(x^2+2)^2}\,dx$$

$$\frac{4-x}{x(x^2+2)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+2} + \frac{Ex+F}{(x^2+2)^2}$$

need trig sub for the $F/(x^2+2)^2$

term!