L'Hopital's Rule

- Tool for computing limits that are "indeterminate": expressions that can't be assigned a consistent numerical value.

Thm: Suppose that $f(x), g(x)$ are functions differentiable near a, $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$ or $\lim_{x \to a} f(x) = \pm\infty$ and $\lim_{x \to a} g(x) = \pm\infty$, and $g'(x) \neq 0$ (except possibly at a). Then

$$\boxed{\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}, \text{ provided RHS exists, or is infinite}}$$

Note: we allow $a = \pm\infty$.

Indeterminate Form	How to Handle
$\frac{0}{0}$ $\frac{\infty}{\infty}$	L'Hopital
$0 \cdot \infty$	Rewrite as $0/0$ or ∞/∞.
$\infty - \infty$	Algebra trick
0^0 1^∞ ∞^0	take log, rewrite as $0/0$ or ∞/∞

Ex: $\displaystyle\lim_{x\to\infty} \frac{\ln(x)}{\sqrt{x}} = \frac{\infty}{\infty}$

$\displaystyle\overset{L'H}{=} \lim_{x\to\infty} \frac{1/x}{1/2\sqrt{x}} = \lim_{x\to\infty} \frac{2}{\sqrt{x}} = 0.$

Ex: $\displaystyle\lim_{x\to\infty} \frac{x^2}{e^x} = \frac{\infty}{\infty} \overset{L'H}{=}$

$\displaystyle\lim_{x\to\infty} \frac{2x}{e^x} = \frac{\infty}{\infty} \overset{L'H}{=} \lim_{x\to\infty} \frac{2}{e^x} = 0.$

Ex: $\displaystyle\lim \sqrt{x}\ln(x) = 0\cdot\infty$

$x \to 0^+$

$$\lim_{x \to 0^+} \sqrt{x}\ln(x) = \lim_{x \to 0^+} \frac{\ln(x)}{1/\sqrt{x}} = \frac{-\infty}{\infty}$$

$$\overset{L'H}{=} \lim_{x \to 0^+} \frac{1/x}{-\frac{1}{2}x^{-3/2}} = \lim_{x \to 0^+} -2\sqrt{x} = 0.$$

<u>Ex:</u> $\lim_{x \to \infty} (1+1/x)^x = 1^\infty$

$L = \lim_{x \to \infty} (1+1/x)^x$ then

$$\ln(L) = \lim_{x \to \infty} x\ln(1+1/x) = \infty \cdot 0$$

$$\ln(L) = \lim_{x\to\infty} \frac{\ln(1+1/x)}{1/x} \Rightarrow \frac{0}{0}$$

L'H

$$= \lim_{x\to\infty} \frac{\frac{1}{1+1/x} \cdot -1/x^2}{-1/x^2} = \lim_{x\to\infty} \frac{1}{1+1/x} = 1$$

So $\ln(L) = 1 \Rightarrow L = e.$

Ex: $\lim_{x\to 4} \frac{1}{\sqrt{x}-2} - \frac{4}{x-4} = \infty - \infty$

$$\lim_{x\to 4} \frac{(x-4) - 4(\sqrt{x}-2)}{(x-4)(\sqrt{x}-2)}$$

Note:

$x-4 = (\sqrt{x}-2)(\sqrt{x}+2)$

$$= \lim_{x \to 4} \frac{(\sqrt{x}-2)\left[(\sqrt{x}+2)-4\right]}{(x-4)(\sqrt{x}-2)} = \lim_{x \to 4} \frac{\sqrt{x}-2}{x-4}$$

$$= 0/0 \overset{L'H}{=} \lim_{x \to 4} \frac{\frac{1}{2\sqrt{x}}}{1} = 1/4.$$

These types are usually quite hard!