Finding Inverses

$f(x) = y$ need g w/

$g(y) = x$.

Ex: $f(x) = (2x+8)^3$.

$(2x+8)^3 = y$ $\qquad \frac{y^{1/3} - 8}{2} = x$

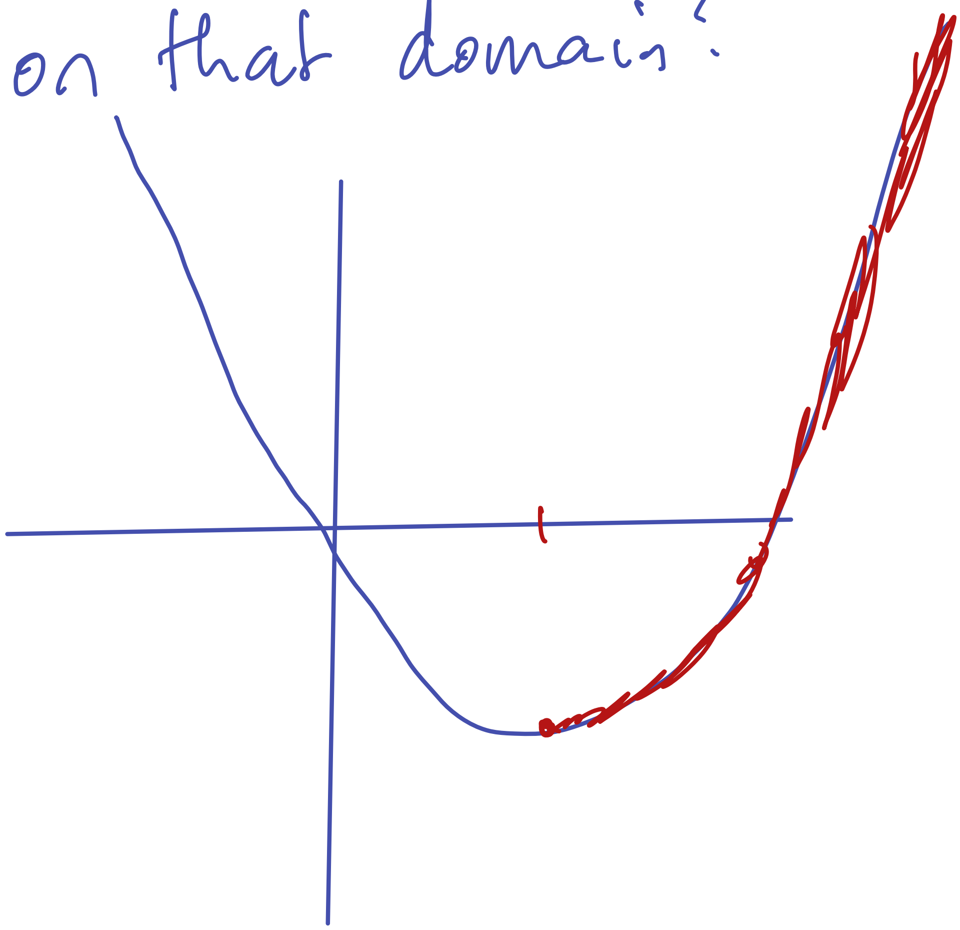
$g(y) = \frac{1}{2}(y^{1/3} - 8)$ Usually

write inverse as function of x,

So $f^{-1}(x) = \frac{1}{2}(x^{1/3} - 8)$.

Ex: $f(x) = x^2 - 2x$.

What's largest domain where f is invertible and formula for inverse on that domain?



need to pick either left/right half of

parabola, otherwise won't be one-to-one if pick larger portion.

Right branch:

domain: $[1, \infty)$

range: $[-1, \infty)$

$x^2 - 2x = y$ $\qquad$ $x^2 - 2x - y = 0$

quadratic formula:

$$x = \frac{2 \pm \sqrt{4+4y}}{2}$$

$$= 1 \pm \sqrt{1+y}$$

which branch to pick?

domain(f^{-1}): $[-1, \infty)$

range(f^{-1}): $[1, \infty)$

means f^{-1} is always pos.

only $1+\sqrt{1+y}$ is always pos. on

$[-1,\infty)$, so $f^{-1}(x) = 1+\sqrt{1+x}$.

Calculus and inverses

One useful fact: Strictly inc/dec functions are always one-to-one (easy to see w/ picture). You know how to check this with derivatives!

What's derivative of inverse?

Since $f(f^{-1}(x)) = x$, chain rule says

$$f'(f^{-1}(x)) \cdot (f^{-1}(x))' = 1, \text{ so } \boxed{(f^{-1}(x))' = \frac{1}{f'(f^{-1}(x))}}$$

This will be key in the next few sections.

Note: don't need to actually know formulas to apply this!

Ex: $f(x) = x^7 + x + 1$. What is $(f^{-1})'(3)$?

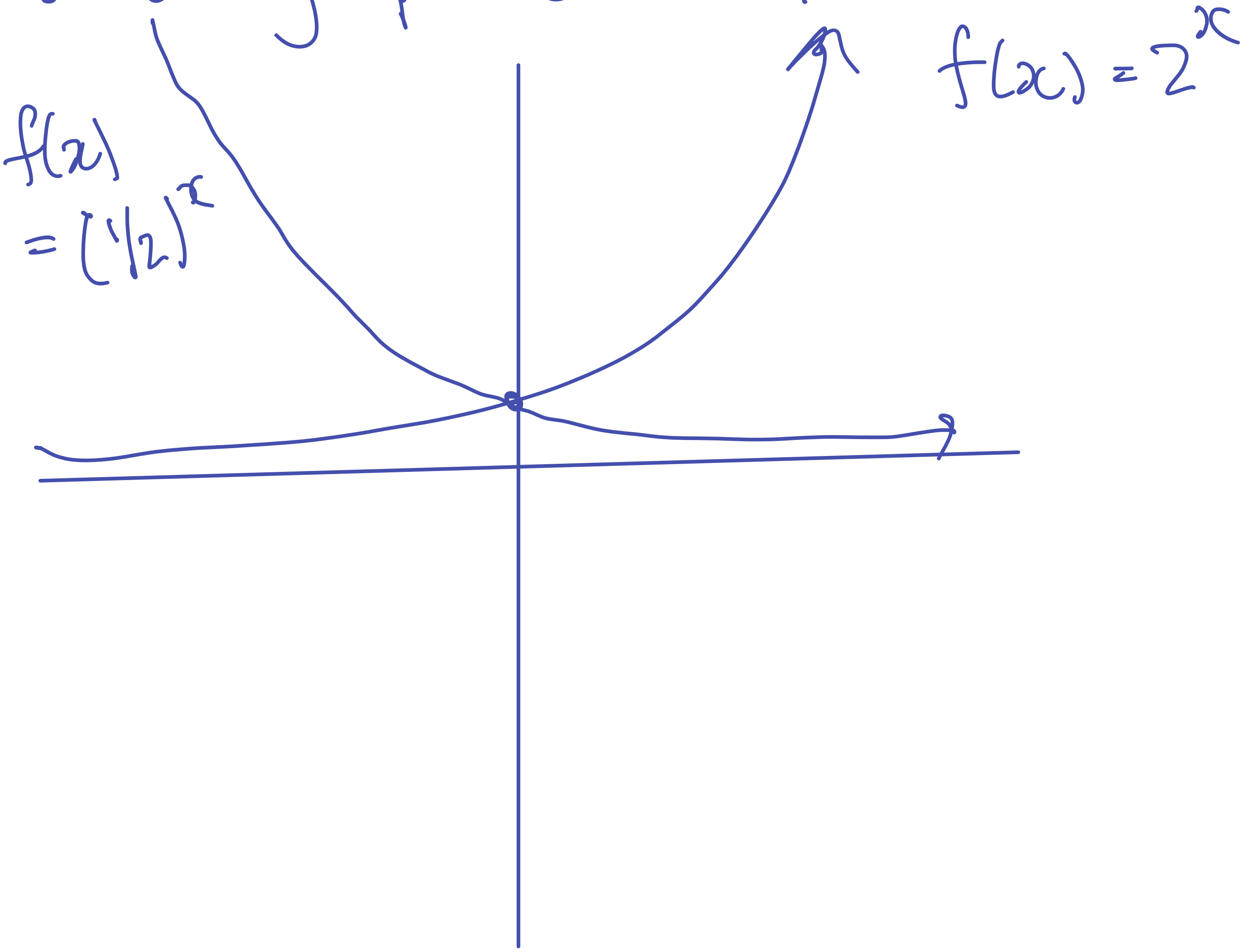
$$(f^{-1})'(3) = \frac{1}{f'(f^{-1}(3))} = \frac{1}{f'(1)} = \frac{1}{7 \cdot 1^6 + 1} = \frac{1}{8}.$$

Exponentials and logarithms:

$f: \mathbb{R} \longrightarrow \mathbb{R}_{>0}$

$f(x) = b^x$

- growth if $b > 1$
- decay if $0 < b < 1$



derivative: $\dfrac{f(x+h)-f(x)}{h} = b^x \dfrac{b^h-1}{h}$

$\lim_{h\to 0} b^x \dfrac{b^h-1}{h} = b^x \cdot \underbrace{C(b)}_{\text{some number}}$, i.e. derivative is prop. to b^x.

In calculus, we mainly look at e^x.

Why? Fact: $\lim_{h\to 0} \dfrac{e^h-1}{h} = 1$ so

$$\boxed{\frac{d}{dx} e^x = e^x.} \qquad \boxed{\int e^x \, dx = e^x + C}$$

Exponentials are invertible with inverses called logarithms:

- $\log_b(b^x) = x \qquad b^{\log_b(x)} = x.$

Exp Rules | Log Rules

- $b^0 = 1 \qquad \log_b(1) = 0$

$b^{x+y} = b^x b^y$

$(b^x)^y = b^{xy}$

$b^{-x} = \frac{1}{b^x}$

$\log_b(xy) = \log_b(x) + \log_b(y)$

$\log_b(x/y) = \log_b(x) - \log_b(y)$

$\log_b(x^a) = a \log_b(x)$

- $\ln(x) = \log_e(x)$
- $\log_b(x) = \ln(x)/\ln(b)$, i.e. all logs differ by a <u>constant</u>. Only need to consider $\ln(x)$ for calculus, then.

$f(x) = e^x \quad f^{-1}(x) = \ln(x)$

using formula,

$$\boxed{\begin{aligned} \frac{d}{dx} \ln(x) &= 1/x \\ \int \frac{1}{x}\, dx &= \ln|x| + C \end{aligned}}$$

Why absolute value? So we can define an antider. on largest domain possible.

Can figure out what $c(b)$ is in der. of b^x is using inv. der. formula

$$f(x) = \log_b(x) \qquad f^{-1}(x) = b^x$$

$$\boxed{\begin{aligned} &\frac{d}{dx} b^x = b^x \ln(b) \qquad \int b^x\,dx = \frac{1}{\ln(b)} b^x + C \\ &\frac{d}{dx} \log_b(x) = \frac{1}{\ln(b)x} \end{aligned}}$$

<u>Ex:</u> $\frac{d}{dx} \ln(x^2)e^{x^2}$

$$= \frac{d}{dx} 2\ln(x)e^{x^2} = \frac{2}{x}e^{x^2} + 4x\ln(x)e^{x^2}$$

<u>Ex:</u> $\int \frac{x}{x^2+2}\,dx \qquad u = x^2+2$

$du = \frac{1}{2}x\,dx$

$$= \int \frac{1}{2}\frac{1}{u}\,du = \frac{1}{2}\ln|u| + C =$$

$$\frac{1}{2}\ln|x^2+2| + C.$$

Ex: $$\frac{d}{dx}\pi^{5x-2} = 5\ln(\pi)\,\pi^{5x-2}$$

Ex: $$\int x\,3^{x^2}dx = \int \frac{1}{2}\,3^u\,du = \frac{1}{2\ln(3)}3^{x^2} + C$$