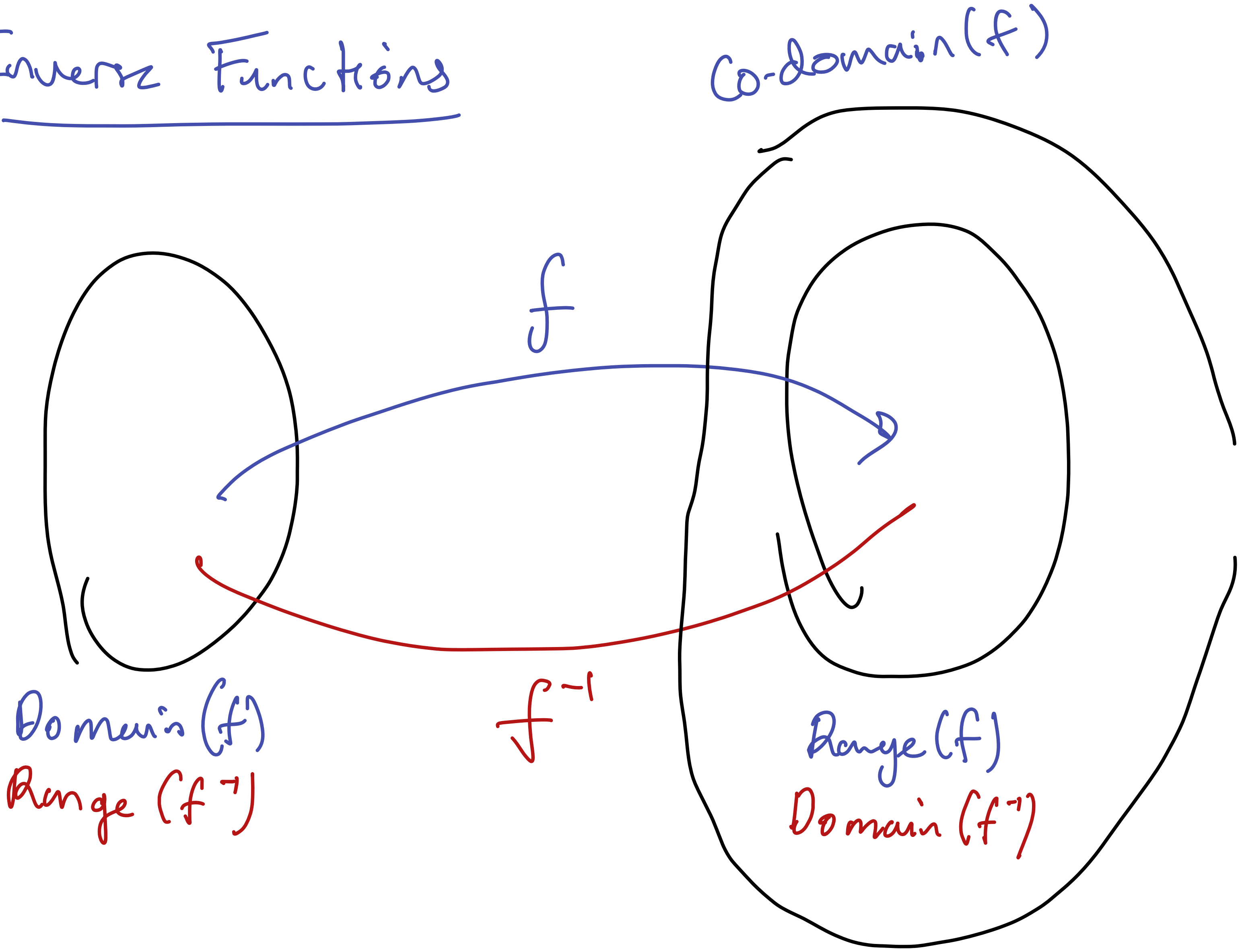
Inverse Functions



Formal def: a function $f: \overset{\text{domain}}{X} \to \overset{\text{co-domain}}{Y}$ is called **invertible** if there is a function $g: Y \to X$ s.t.

- $g(f(x)) = x$ for all $x \in X$
- $f(g(y)) = y$ for all $y \in Y$.

If an inverse exists, we usually write

$g = f^{-1}$.

Intuitively, f^{-1} undoes f. We make the following observations:

1. For any $y \in Y$, $f(f^{-1}(y)) = y$ so with $x = f^{-1}(y)$, $f(x) = y$.

Every value in co-domain gets mapped to

2. If $f(x) = f(y)$, $f^{-1}(f(x)) = x = y = f^{-1}(f(y))$

distinct values map to distinct values

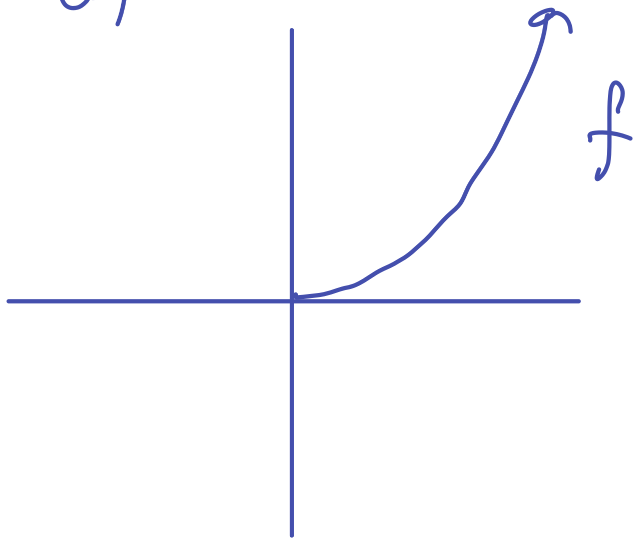
The first condition is called <u>onto</u> and the second is <u>one-to-one</u>.

<u>Equivalent def:</u> f is invertible $\iff$ f is one-to-one and onto.

Ex: $f: \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is invertible

$f(x) = x^2$

w/



$f^{-1}: \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$

$f^{-1}(x) = \sqrt{x}$

Ex: $f: \mathbb{R}_{\geq 0} \to \mathbb{R}$ is not onto, so it's not invertible.

$f(x) = x^2$

Ex: $f: \mathbb{R} \to \mathbb{R}_{\geq 0}$ is not one-to-one so it's not invertible.

$f(x) = x^2$

Ex: $f: \mathbb{R} \to \mathbb{R}$ is neither onto

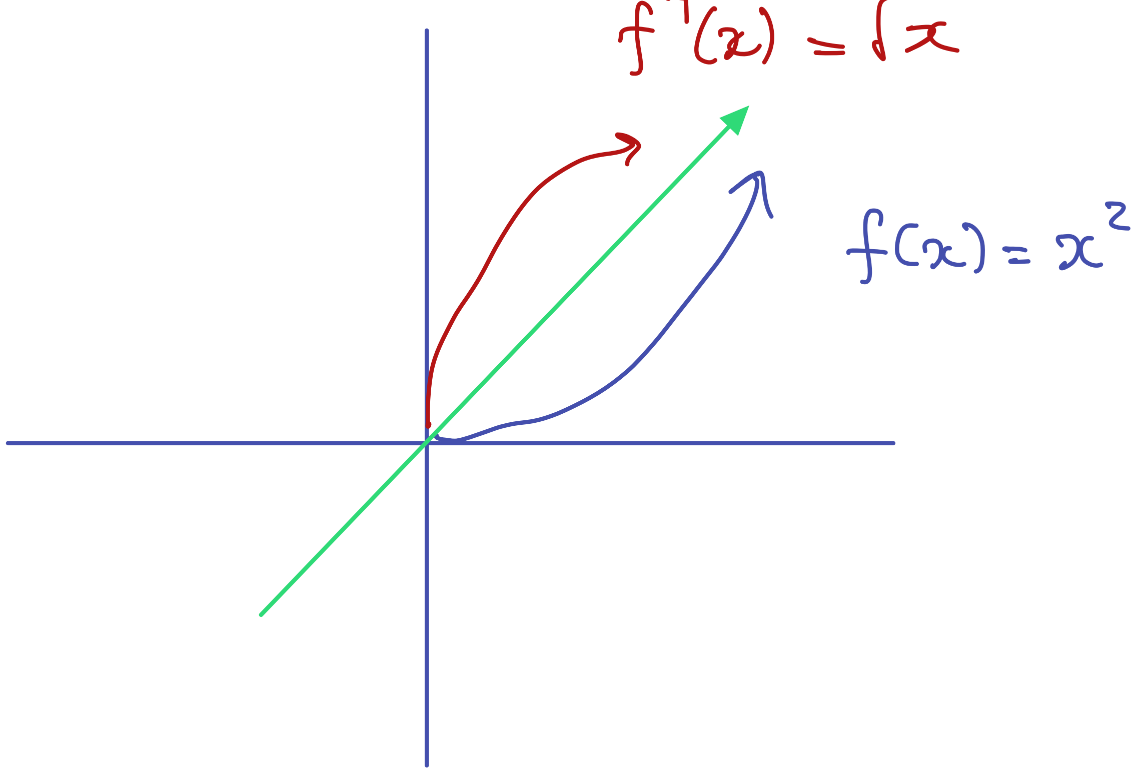
$f(x) = x^2$ nor one-to-one.

Point: domain/co-domain matter for the function, not just the formula! Must have domain(f) = co-domain(f) to be invertible.

Geometric interpretation

Saying $f(x) = y$ means (x, y) is a point on graph of f.

Since $f^{-1}(y) = x$ this means (y, x) is a point on graph of f^{-1}.



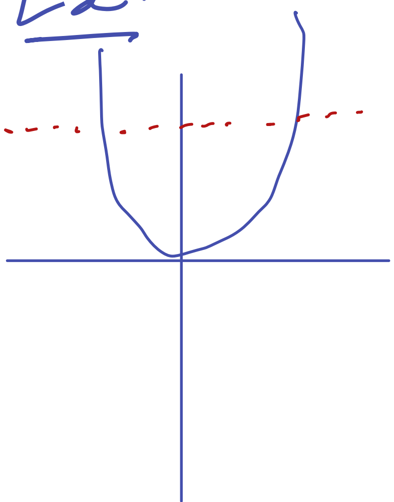
geometrically, this is a reflection around the line $y = x$. This lets us graph the inverse without knowing a formula for f!

Can also check invertibility from graph:

- if graph intersects a hor. line in $\geq$ two places, means f is not one-to-one.

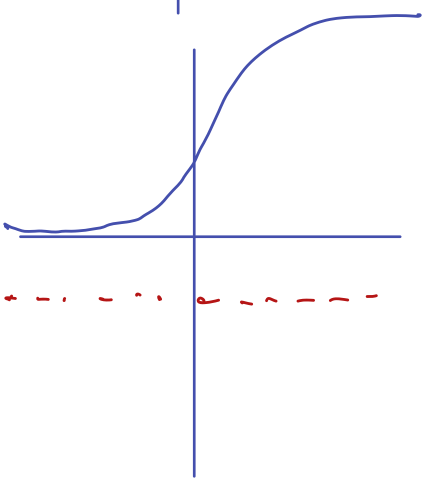
- if there is a hor. line f never hits (for y-value in codomain) then f is not onto.

<u>Ex:</u>



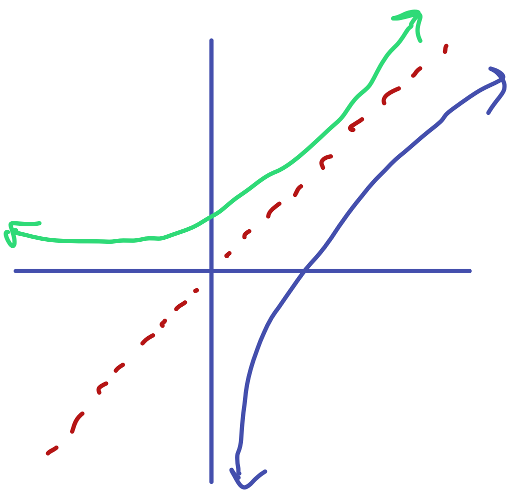
$f: \mathbb{R} \to \mathbb{R}_{\geq 0}$

nor one-to-one



$f: \mathbb{R} \to \mathbb{R}$

nor onto



$f: \mathbb{R}_{>0} \to \mathbb{R}$

$f^{-1}: \mathbb{R} \to \mathbb{R}_{>0}$

Algebraically finding inverse

Since $f(x) = y$ then we need to find g w/ $g(y) = x$

Ex: $f: \mathbb{R} \to \mathbb{R}$

$$f(x) = (2x+8)^3$$

Set $y = (2x+8)^3$. Solve for x:

$$\frac{y^{1/3} - 8}{2} = x, \quad \text{so} \quad g(y) = \frac{1}{2}\left(y^{1/3} - 8\right).$$

Usually write functions as functions of x, so $f^{-1}(x) = \frac{1}{2}\left(x^{1/3} - 8\right)$.

Ex: $f(x) = x^2 - 2x$.

largest domain where invertible: $[1, \infty)$

$y = x^2 - 2x$ $\quad$ $x^2 - 2x - y = 0$

$$x = \frac{2 \pm \sqrt{4+4y}}{2} = 1 \pm \sqrt{1+y}$$

which branch is inverse?

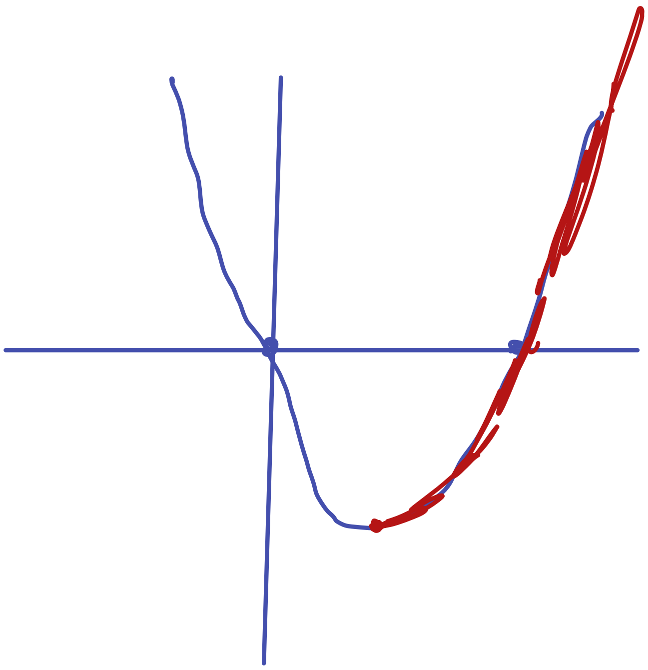
($(-\infty, 1]$ also works)

Range on this domain: $[-1, \infty)$

domain of f^{-1}: $[-1, \infty)$

Range of f^{-1}: $[1, \infty)$

need to pick pos. branch for right range



$f^{-1}(x) = 1 + \sqrt{1+x}$