Selected Solutions to Homework 7

Tim Smits

March 17, 2023

11.6.12 Find the interval of convergence of $\sum_{n=0}^{\infty}(-1)^n \frac{n}{4^n} x^{2n}$.

Solution: Running the ratio test, the series converges as long as $\lim_{n\to\infty} \frac{4^n(n+1)}{4^{n+1}n}|x|^2 < 1$. Computing the limit, we require $\frac{1}{4}|x|^2 < 1$, so $|x| < 2$. This says $R = 2$ so the series converges inside $(-2, 2)$. At $x = 2$ the series is $\sum_{n=0}^{\infty}(-1)^n n$ which diverges, and at $x = -2$ the series is $\sum_{n=0}^{\infty} n$ which also diverges. Therefore, the interval of convergence is $(-2, 2)$.

11.6.38 Find a power series expansion for $\frac{1}{4+3x}$ centered at $c = 0$.

Solution: We have $\frac{1}{4+3x} = \frac{1}{4}\frac{1}{1-(-3x/4)} = \frac{1}{4}\sum_{n=0}^{\infty}(-3x/4)^n = \sum_{n=0}^{\infty}(-1)^n \frac{3^n x^n}{4^{n+1}}$.

11.6.42 Find a power series expansion for $\frac{4x^3}{(1-x^4)^2}$ centered at 0.

Solution: We have $\frac{1}{1-x^4} = \sum_{n=0}^{\infty} x^{4n}$ so differentiating yields $\frac{4x^3}{(1-x^4)^2} = \sum_{n=1}^{\infty} 4nx^{4n-1}$.

11.6.46 Expand $\frac{1}{1-x}$ into a power series with centers $c = 2$ and $c = -2$.

Solution: We have $\frac{1}{1-x} = \frac{1}{-1-(x-2)} = -\frac{1}{1-(-(x-2))} = -\sum_{n=0}^{\infty}(-1)^n(x-2)^n = \sum_{n=0}^{\infty}(-1)^{n+1}(x-2)^n$, which is the power series expansion centered at 2. Similarly, we have $\frac{1}{1-x} = \frac{1}{3-(x+2)} = \frac{1}{3}\frac{1}{1-((x+2)/3)} = \frac{1}{3}\sum_{n=0}^{\infty}(\frac{x+2}{3})^n = \sum_{n=0}^{\infty}\frac{1}{3^{n+1}}(x+2)^n$, which is the power series expansion centered at $c = -2$.

11.8.12 Find the Maclaurin series and interval of convergence for $f(x) = x^2 e^{x^2}$.

Solution: We have $e^{x^2} = \sum_{n=0}^{\infty}\frac{x^{2n}}{n!}$ so $x^2 e^{x^2} = \sum_{n=0}^{\infty}\frac{x^{2n+2}}{n!}$. The interval of convergence is easily seen to be $(-\infty, \infty)$.

11.8.14 Find the Maclaurin series and interval of convergence for $f(x) = \frac{1-\cos(x)}{x}$.

Solution: We have $\cos(x) = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$, so $1 - \cos(x) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}x^{2n}}{(2n)!}$. Dividing by x gives $\frac{1-\cos(x)}{x} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}x^{2n-1}}{(2n)!}$.

11.8.20 Find the 4th order Taylor polynomial for $e^x \ln(1-x)$ centered at $c = 0$.

Solution: We have $e^x = 1 + x + 1/2x^2 + 1/6x^3 + 1/24x^4 + \ldots$ and $\ln(1-x) = -x - x^2/2 - x^3/3 - x^4/4 - \ldots$. We multiply these together and get rid of the terms of degree > 4. Doing this, you'll find the polynomial we are looking for to be $-x - \frac{3x^2}{2} - \frac{4x^3}{3} - x^4$.

Answers to even non-graded problems

11.6.22 : $[-1, 1)$

11.6.28 : $(2/3, 4/3)$

11.8.16 : $\sum_{n=0}^{\infty} \frac{x^{n+2}}{n!} + \sum_{n=0}^{\infty} \frac{2x^{n+1}}{n!}$