Selected Solutions to Homework 5

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February 24, 2023

11.1.60 Compute $\lim_{n\to\infty} \frac{(-1)^n n^3+2^{-n}}{3n^3+4^{-n}}$

Solution: The sequence is alternating between positive and negative terms. Consider the positive part $\frac{n^3+2^{-n}}{3n^3+4^{-n}}$ and the negative part $\frac{-n^3+2^{-n}}{3n^3+4^{-n}}$. The first sequence tends to $\frac{1}{3}$ as $n \to \infty$, and the second sequence tends to $-\frac{1}{3}$ as $n \to \infty$ by the standard method of dividing numerator/denominator by n^3. Since the two sequences tend to different limits, $\lim_{n\to\infty} \frac{(-1)^n n^3+2^{-n}}{3n^3+4^{-n}}$ does not exist.

11.2.13 Compute $\sum_{n=1}^{\infty} \frac{1}{4n^2-1}$ along with S_3, S_4, S_5.

Solution: Using the identity $\frac{1}{4n^2-1} = \frac{1}{2}(\frac{1}{2n-1} - \frac{1}{2n+1})$ we have $\sum_{n=1}^{\infty} = \sum_{n=1}^{\infty} \frac{1}{2}(\frac{1}{2n-1} - \frac{1}{2n+1}) = \lim_{N\to\infty} \sum_{n=1}^{N} \frac{1}{2}(\frac{1}{2n-1} - \frac{1}{2n+1})$. Since this expression is a telescoping series, the N-th partial sum is $\frac{1}{2}(1 - \frac{1}{2N+1})$. This gives $S_3 = \frac{1}{2}(1-1/7)$, $S_4 = \frac{1}{2}(1-1/9)$ and $S_5 = \frac{1}{2}(1-1/11)$. The value of the infinite sum is $\lim_{N\to\infty} S_N = \frac{1}{2}$.

11.3.25 Determine the convergence or divergence of $\sum_{n=1}^{\infty} \frac{2}{3^n+3^{-n}}$.

Solution: Since $3^n + 3^{-n} \geq 3^n$, we have $\frac{2}{3^n+3^{-n}} \leq 2(\frac{1}{3})^n$. Since $\sum_{n=1}^{\infty} 2(\frac{1}{3})^n$ is a convergent geometric series, by the direct comparison test we see that $\sum_{n=1}^{\infty} \frac{2}{3^n+3^{-n}}$ converges.

11.3.41 Determine the convergence or divergence of $\sum_{n=1}^{\infty} \frac{3n+5}{n(n-1)(n-2)}$.

Solution: As $n \to \infty$, we see that $\frac{3n+5}{n(n-1)(n-2)} \approx \frac{3n}{n^3} = \frac{3}{n^2}$. Run the limit comparison test with $a_n = \frac{3n+5}{n(n-1)(n-2)}$ and $b_n = \frac{3}{n^2}$: we have $\frac{a_n}{b_n} = \frac{3n^3+5n^2}{3n(n-1)(n-2)} \to 1$ using standard methods. Since $\sum_{n=1}^{\infty} \frac{3}{n^2}$ is a convergent p-series, by the limit comparison test, $\sum_{n=1}^{\infty} \frac{3n+5}{n(n-1)(n-2)}$ converges.

11.3.59 Determine the convergence or divergence of $\sum_{n=2}^{\infty} \frac{1}{\ln(n)^4}$.

Solution: Since $\ln(n) \leq n^a$ for any $a > 0$ eventually, we have $\ln(n)^4 \leq n^{4a}$ for any $a > 0$ eventually, and therefore $\frac{1}{\ln(n)^4} \geq \frac{1}{n^{4a}}$ for any $a > 0$ eventually. Picking $a = \frac{1}{8}$, we see that eventually, $\frac{1}{\ln(n)^4} \geq \frac{1}{\sqrt{n}}$. Since $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$ is a divergent p-series, by the direct comparison test, we see that $\sum_{n=2}^{\infty} \frac{1}{\ln(n)^4}$ diverges.

11.3.68 Determine the convergence or divergence of $\sum_{n=1}^{\infty} \frac{\sin(1/n)}{\sqrt{n}}$.

Solution: As $n \to \infty$, $\sin(1/n) \approx \frac{1}{n}$ and so $\frac{\sin(1/n)}{\sqrt{n}} \approx \frac{1/n}{\sqrt{n}} = \frac{1}{n^{3/2}}$. Run the limit comparison test with $a_n = \frac{\sin(1/n)}{\sqrt{n}}$ and $b_n = \frac{1}{n^{3/2}}$: we have $\frac{a_n}{b_n} = n\sin(1/n) = \frac{\sin(1/n)}{1/n}$ and $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{u\to 0} \frac{\sin(u)}{u} = 1$ with $u = \frac{1}{n}$. Since $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges, by the limit comparison test we see that $\sum_{n=1}^{\infty} \frac{\sin(1/n)}{\sqrt{n}}$ also converges.

11.3.71 Determine the convergence or divergence of $\sum_{n=4}^{\infty} \frac{\ln(n)}{n^2-3n}$.

Solution: As $n \to \infty$, we have $\frac{\ln(n)}{n^2-3n} \approx \frac{\ln(n)}{n^2}$. Do the limit comparison test with $a_n = \frac{\ln(n)}{n^2-3n}$ and $b_n = \frac{\ln(n)}{n^2}$ to see that $\sum_{n=4}^{\infty} \frac{\ln(n)}{n^2-3n}$ and $\sum_{n=4}^{\infty} \frac{\ln(n)}{n^2}$ have the same behavior. To see what $\sum_{n=4}^{\infty} \frac{\ln(n)}{n^2}$ does, note that eventually, we must have $\ln(n) \leq \sqrt{n}$ so that $\frac{\ln(n)}{n^2} \leq \frac{1}{n^{3/2}}$. Since $\sum_{n=4}^{\infty} \frac{1}{n^{3/2}}$ is a convergent p-series, by the direct comparison test, $\sum_{n=4}^{\infty} \frac{\ln(n)}{n^2}$ converges, and we are done.

11.3.75 Determine the convergence or divergence of $\sum_{n=2}^{\infty} \frac{4n^2+15n}{3n^4-5n^2-17}$.

Solution: As $n \to \infty$, $\frac{4n^2+15n}{3n^4-5n^2-17} \approx \frac{4n^2}{3n^4} = \frac{4}{3n^2}$. Do the limit comparison test with $a_n = \frac{4n^2+15n}{3n^4-5n^2-17}$ and $b_n = \frac{4}{3n^2}$: then $\frac{a_n}{b_n} = \frac{12n^4+45n^2}{12n^4-20n^2-68}$ and it's a standard exercise to see that $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$. Since $\sum_{n=2}^{\infty} \frac{4}{3n^2}$ is a convergent p-series, by the limit comparison test, $\sum_{n=2}^{\infty} \frac{4n^2+15n}{3n^4-5n^2-17}$ converges.

Answers to even non-graded problems

11.1.68 : 0

11.2.22 : Standard limit computation: multiply/divide by conjugate expression.

11.2.34 : $-4/15$

11.2.38 : $125/18$

11.3.34 : Converges

11.3.52 : Converges

11.3.54 : Converges