Selected Solutions to Homework 4

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February 17, 2023

8.7.22 $\int_1^2 \frac{1}{(x-1)^2}\,dx$

Solution: The integral blows up at $x = 1$, so it's an improper integral. Write it as $\lim_{R\to 1}\int_1^R \frac{1}{(x-1)^2}\,dx = \lim_{R\to 1} -\frac{1}{x-1}\Big|_1^R = \lim_{R\to 1}\frac{1}{R-1} - 1 = \infty$, so the integral diverges.

8.7.48 $\int_{-\infty}^{\infty} \frac{1}{(x^2+1)^{3/2}}\,dx$

Solution: This is a doubly improper integral, so it needs to be split up into two separate integrals in order to evaluate it. We'll split it up at 0, because by symmetry, $\int_{-\infty}^0 \frac{1}{(x^2+1)^{3/2}}\,dx$ and $\int_0^\infty \frac{1}{(x^2+1)^{3/2}}\,dx$ must have the same value.

To compute $\int_0^\infty \frac{1}{(x^2+1)^{3/2}}\,dx$, write it as $\lim_{R\to\infty}\int_0^R \frac{1}{(x^2+1)^{3/2}}\,dx$. The integral here is a standard trig sub exercise, the resulting limit is $\lim_{R\to\infty}\frac{x}{\sqrt{x^2+1}} = \lim_{R\to\infty}\frac{1}{\sqrt{1+1/x^2}} = 1$. Therefore, the integral will converge, to a value of 2.

8.7.62 $\int_1^\infty \frac{1}{(x^3+2x+4)^{1/2}}\,dx$

Solution: As $x \to \infty$, $x^3 + 2x + 4 \approx x^3$, so the our integral and $\int_1^\infty \frac{1}{x^{3/2}}\,dx$ should have the same behavior. The latter converges, so let's try and prove our integral converges with a comparison. Certainly, we have $x^3+2x+4 \geq x^3$ on $[1,\infty)$, so inverting and taking a square root yields $\frac{1}{(x^3+2x+4)^{1/2}} \leq \frac{1}{x^{3/2}}$. Therefore, by the comparison test, $\int_1^\infty \frac{1}{(x^3+2x+4)^{1/2}}\,dx$ converges.

8.7.74 $\int_1^\infty \frac{1}{(x+x^2)^{1/3}}\,dx$

Solution: As $x \to \infty$, $\frac{1}{(x+x^2)^{1/3}} \approx \frac{1}{x^{2/3}}$, so our integral should behave like $\int_1^\infty \frac{1}{x^{2/3}}\,dx$, which diverges. To get an inequality going the correct way, note that on $[1,\infty)$ we have $(x+x^2)^{1/3} \leq (x^2+x^2)^{1/3} = 2^{1/3}x^{2/3}$, so inverting yields $\frac{1}{(x+x^2)^{1/3}} \geq \frac{1}{2^{1/3}x^{2/3}}$. Integrating then shows that our integral diverges by the comparison test.

8.7.77 $\int_0^\infty \frac{1}{x^{1/2}(x+1)}\,dx$

Solution: The integral is improper for two reasons: it blows up at $x = 0$, and it has an infinite limit. Therefore, we have to split it up into two easier to understand improper integrals to determine what happens. We'll split it up at $x = 1$: write $\int_0^\infty \frac{1}{x^{1/2}(x+1)}\,dx = \int_0^1 \frac{1}{x^{1/2}(x+1)}\,dx + \int_1^\infty \frac{1}{x^{1/2}(x+1)}\,dx$. For the first integral, near $x = 0$ the integrand looks like $\frac{1}{x^{1/2}}$, so that piece should converge. To prove it, note $x^{1/2}(x+1) \geq x^{1/2}$ on $[0,1]$, so $\frac{1}{x^{1/2}(x+1)} \leq \frac{1}{x^{1/2}}$, and $\int_0^1 \frac{1}{x^{1/2}}\,dx$ converges. For the second piece, as $x \to \infty$ we have $\frac{1}{x^{1/2}(x+1)} \approx \frac{1}{x^{3/2}}$, and $\int_1^\infty \frac{1}{x^{3/2}}\,dx$ converges. Explicitly, we have $x^{1/2}(x+1) \geq x^{3/2}$ so inverting gives $\frac{1}{x^{1/2}(x+1)} \leq \frac{1}{x^{3/2}}$, and integrating then shows that $\int_1^\infty \frac{1}{x^{1/2}(x+1)}\,dx$ converges. Since both pieces converge, the original integral therefore converges.

8.8.50 Find N such that S_N approximates $\int_0^4 xe^x\,dx$ to within 10^{-9}.

Solution: Taking four derivatives, we have $|f^{(4)}(x)| = |(x+4)e^x| = (x+4)e^x$. This function is strictly increasing on $[0,4]$, so it's maximal value happens at $x = 4$. Therefore, we may take $K_4 = 8e^4$. Plugging into the error bound, we want to find N such that $\frac{8e^4}{180N^4}4^5 \leq \frac{1}{10^9}$. Solving for N gives $N \approx 1255.52$. Since we need an even N for Simpson's rule, we may take $N = 1256$.

1. Using what we've learned about integration, we know that $\int_0^1 \frac{4}{x^2+1}\,dx = \pi$. This gives us a method for computing π!

(a) Use the error bound formula to find a value of N such that the N-th Simpson's rule approximation S_N is guaranteed to approximate π to within 6 decimal places. *(You can use WolframAlpha to figure out a choice for the constant K_4.)*

(b) For this particular integral, the error estimate is very conservative. Compute S_8 and compare with π. What's the error in the approximation?

Solution:

(a) Using WolframAlpha, we have $|f^{(4)}(x)| = |\frac{(96(1-10x^2+5x^4))}{(1+x^2)^5}|$. On $[0,1]$, this has a maximum value of 96 at $x = 0$, so we may choose $K_4 = 96$. Plugging into the error bound, we need $\frac{96}{180N^4} \leq \frac{1}{10^6}$, and solving then gives $N \approx 27.024$. Since we need even N for Simpson's rule, we may take $N = 28$ to guarantee the desired accuracy.

(b) With $\Delta x = 1/8$, we have $S_8 = \frac{1}{24}(f(0)+4f(1/8)+2f(2/8)+4f(3/8)+2f(4/8)+4f(5/8)+2f(6/8)+4f(7/8)+f(1)) \approx 3.1415925$. This agrees with π to 6 decimal places, and so the actual error is less than $1/10^6$.

2. The *Gamma function*, $\Gamma(x)$, is defined by $\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}\,dt$. The Gamma function is one of the most commonly occurring functions in mathematics, statistics, and physics. One can show that $\Gamma(x)$ converges for $x > 0$ (proving that is a great challenge if you're up for it!)

(a) Use integration by parts to prove that $\Gamma(x+1) = x\Gamma(x)$ for $x > 0$.

(b) Compute $\Gamma(1)$ and then use part (a) to compute $\Gamma(n)$ for $n = 2, 3, 4, 5$. Write down a formula for $\Gamma(n)$ for arbitrary integer $n \geq 1$.

Solution:

(a) Plugging in, we have $\Gamma(x+1) = \int_0^\infty e^{-t}t^x\,dt$. Using integration by parts with $u = t^x$ and $dv = e^{-t}$, we find $\Gamma(x+1) = -e^{-t}t^x|_0^\infty + \int_0^\infty e^{-t}xt^{x-1}\,dt = -e^{-t}t^x|_0^\infty + x\Gamma(x)$. It remains to evaluate the first term. We have $-e^{-t}t^x|_0^\infty = \lim_{R\to\infty} -\frac{R^x}{e^R} = 0$. To see this, you must apply L'Hopital's rule $\lceil x \rceil$ times after which point the exponent in the numerator becomes negative (here $\lceil x \rceil$ means round x up to the nearest integer. If it's not clear why this works, try an example with say, $x = 2.5$). Therefore, $\Gamma(x+1) = x\Gamma(x)$.

(b) We have $\Gamma(1) = \int_0^\infty e^{-t}\, dt = 1$. Using part (b), we then find $\Gamma(2) = 1 \cdot 1 = 1$, $\Gamma(3) = 2 \cdot 1 = 2$, $\Gamma(4) = 3 \cdot 2 = 6$, and $\Gamma(5) = 4 \cdot 6 = 24$. In general, $\Gamma(n) = (n-1)!$.

Note: this means that the Gamma function is an *interpolation* of the factorial sequence: it's a continuous function that agrees with the (shifted) factorial at integer values. There are many other continuous functions that have this property, but the Gamma function is the one that is most commonly used (because of other special properties it has that we won't go into.)

Answers to even non-graded problems

8.7.26 : Converges to $-1/2$

8.7.64 : Converges

8.7.66 : Converges

8.7.68 : Converges

8.8.10 : $T_5 \approx 1.12096$, $M_5 \approx 1.11716$

8.8.18 $S_6 \approx .904523$