Selected Solutions to Homework 3

Tim Smits

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8.2.35 $\int \tan^6(x)\sec^4(x)\,dx$

Solution: Leave a $\sec^2(x)$ left over, so this is $\int \tan^6(x)(1+\tan^2(x))\sec^2(x)\,dx$ and set $u = \tan(x)$ to get $\int u^6(1+u^2)\,du = \frac{1}{7}u^7 + \frac{1}{9}u^9 + C = \frac{1}{7}\tan^7(x) + \frac{1}{9}\tan^9(x) + C$.

8.2.42 $\int \sin^2(x)\sec^4(x)\,dx$

Solution: Write the integral as $\int \frac{\sin^2(x)}{\cos^4(x)}\,dx = \int \frac{1-\cos^2(x)}{\cos^4(x)}\,dx = \int \sec^4(x) - \sec^2(x)\,dx = \int (1+\tan^2(x))\sec^2(x) - \sec^2(x)\,dx = \tan(x) + \frac{1}{3}\tan^3(x) - \tan(x) + C = \frac{1}{3}\tan^3(x) + C$.

8.3.18 $\int \sqrt{12+4t^2}\,dt$

Solution: Write the integral as $2\int \sqrt{3+t^2}\,dt$ and use $t = \sqrt{3}\tan(\theta)$ to get $2\int \sqrt{3}\sec(\theta)\cdot\sqrt{3}\sec^2(\theta)\,d\theta = 6\int \sec^3(\theta)\,d\theta$. We've seen the integral of $\sec^3(\theta)$ computed before, so doing the integration gives $3\sec(\theta)\tan(\theta) + 3\ln|\sec(\theta)+\tan(\theta)| + C$. Since $\tan(\theta) = t/\sqrt{3}$, drawing the correct right triangle yields $\sec(\theta) = \frac{\sqrt{3+t^2}}{\sqrt{3}}$ and so plugging in gives us an answer of $\frac{\sqrt{3+t^2}}{\sqrt{3}} + \frac{t}{\sqrt{3}} + \ln|\frac{\sqrt{3+t^2}}{\sqrt{3}} + \frac{t}{\sqrt{3}}| + C$.

8.3.25 $\int \frac{dz}{z^3\sqrt{z^2-4}}$

Solution: Set $z = 2\sec\theta$, so plugging in gives $\int \frac{2\sec(\theta)\tan(\theta)}{8\sec^3(\theta)2\tan(\theta)}\,d\theta = \frac{1}{8}\int \frac{1}{\sec^2(\theta)}\,d\theta = \frac{1}{8}\int \cos^2(\theta)\,d\theta$. Using the power reduction formula and integrating, we get $\frac{\theta}{16} + \frac{1}{32}\sin(2\theta) + C = \frac{\theta}{16} + \frac{1}{16}\sin(\theta)\cos(\theta) + C$. We have $\sec(\theta) = z/2$, so drawing the correct triangle gives $\sin(\theta) = \frac{\sqrt{z^2-4}}{z}$ and $\cos(\theta) = \frac{2}{z}$ so plugging in gives an answer of $\frac{\sec^{-1}(z/2)}{16} + \frac{1}{16}\frac{2\sqrt{z^2-4}}{z^2} + C$.

8.3.32 $\int \frac{x^2}{(x^2+1)^{3/2}}\,dx$

Solution: Set $x = \tan\theta$, so the integral becomes $\int \frac{\tan^2(\theta)}{\sec^3(\theta)}\sec^2(\theta)\,d\theta = \int \sin^2(\theta)\cos(\theta)\,d\theta = \frac{1}{3}\sin^3(\theta) + C$. Drawing the correct triangle gives $\sin(\theta) = \frac{x}{\sqrt{1+x^2}}$ so we get an answer of $(\frac{x}{\sqrt{1+x^2}})^3 + C$.

8.5.24 $\int \frac{dx}{(x-4)^2(x-1)}\,dx$

Solution: Write $\frac{1}{(x-4)^2(x-1)} = \frac{A}{x-4} + \frac{B}{(x-4)^2} + \frac{C}{x-1}$. Multiplying through, $1 = A(x-4)(x-1) + B(x-1) + C(x-4)^2$. Plugging in $x = 1$ gives $C = \frac{1}{9}$ and plugging in $x = 4$ gives $B = \frac{1}{3}$. Plugging in $x = 0$ says $1 = 4A - B + 16C$ so that $A = -\frac{1}{9}$. We then need to integrate $\frac{-1/9}{x-4} + \frac{1/3}{(x-4)^2} + \frac{1/9}{x-1}$ which simply integrates to $-\frac{1}{9}\ln|x-4| - \frac{1}{3(x-4)} + \frac{1}{9}\ln|x-1| + C$.

8.5.34 $\int \frac{x^2}{(x+1)(x^2+1)}\,dx$

Solution: Write $\frac{x^2}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$. Multiplying through, $x^2 = A(x^2+1) + (Bx+C)(x+1)$. Expanding and collecting coefficients, we get $x^2 = (A+B)x^2 + (B+C)x + (A+C)$. This means $A+B = 1, B+C = 0, A+C = 0$. Using $B = -C$, this means $A - C = 1$ and $A + C = 0$ so $A = 1, B = 1, C = -1$. We then need to integrate $\frac{1}{x+1} + \frac{x-1}{x^2+1}$ which is easy, and integrates to $\ln|x+1| + \frac{1}{2}\ln|x^2+1| - \tan^{-1}(x) + C$.

8.5.51 $\int \frac{\sqrt{x}}{x-1}\,dx$

Solution: Set $u = \sqrt{x}$ so $dx = 2u\,du$. Plugging in, we get $\int \frac{2u^2}{u^2-1}\,du = \int \frac{2u^2-2+2}{u^2-1}\,du = \int 2 + \frac{2}{(u-1)(u+1)}\,du$. Write $\frac{2}{(u-1)(u+1)} = \frac{A}{u-1} + \frac{B}{u+1}$ and multiply through to get $2 = A(u+1) + B(u-1)$. It's clear from this that $A = 1$ and $B = -1$, so we get $\int 2 + \frac{2}{(u-1)(u+1)}\,du = 2u + \ln|u-1| - \ln|u+1| + C = 2\sqrt{x} + \ln|\sqrt{x}-1| - \ln|\sqrt{x}+1| + C$.

Answers to even non-graded problems

8.2.26 : $\frac{1}{501}\cos^{501}(y) - \frac{1}{499}\cos^{499}(y) + C$

8.2.30 : $-\frac{1}{2}\cot(x)\csc(x) + \frac{1}{2}\ln|\csc(x) - \cot(x)| + C$

8.2.36 : $\frac{1}{4}\tan(x)\sec^3(x) - \frac{1}{8}\tan(x)\sec(x) - \frac{1}{8}\ln|\sec(x) + \tan(x)| + C$

8.3.14 : (a): $\frac{t}{\sqrt{t^2+1}} + C$. (b): $-\frac{1}{\sqrt{t^2+1}} + C$.

8.3.22 : $\frac{1}{5}(9 - x^2)^{5/2} - 3(9 - x^2)^{3/2} + C$

8.5.8 : $\frac{1}{2}x^2 - \frac{1}{2}\ln(x^2 + 1) + \tan^{-1}(x) + C$

8.5.16 : $3\ln|x| - 3\ln|x + 1| + \frac{3}{x+1} + C$

2. Many different approaches. One possible set of ways: (a): parts with $u = x$ and $dv = \sec^2(x)$(b): trig sub with $2x = \sec(\theta)$ (c): u-sub with $u = \cos(x)$ (d): partial fractions (e): u-sub with $u = \sin(2x)$ (f): u-sub with $u = x + 1$.