Selected Solutions to Homework 2

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7.5.48 $\lim_{x\to 0^+} x^{\sin(x)}$

Solution: This is a type 0^0 limit. Set $L = \lim_{x\to 0^+} x^{\sin(x)}$, so that $\ln(L) = \lim_{x\to 0^+} \sin(x)\ln(x)$, which is now type $0 \cdot -\infty$. Rewrite as $\lim_{x\to 0^+} \frac{\ln(x)}{\csc x}$. This is indeterminate of type $\frac{-\infty}{\infty}$, so apply L'Hopital's rule to get $\lim_{x\to 0^+} \frac{1/x}{-\csc(x)\cot(x)} = \lim_{x\to 0^+} \frac{\sin^2(x)}{x\cos(x)}$. Applying L'Hopital's rule again, this is $\lim_{x\to 0^+} \frac{2\sin(x)\cos(x)}{\cos(x) - x\sin(x)} = 0$. Thus, $\ln(L) = 0$ so $L = 1$.

7.6.60 $\int_0^4 \frac{dt}{4t^2+9}$

Solution: $\int_0^4 \frac{dt}{4t^2+9} = \frac{1}{9}\int_0^4 \frac{dt}{(\frac{2}{3}t)^2+1}$. Set $u = \frac{2}{3}t$, so $du = \frac{2}{3}\,dt$. Subbing in, this becomes $\frac{1}{9}\int_0^{8/3} \frac{1}{1+u^2}\frac{3}{2}\,du = \frac{1}{6}\tan^{-1}(u)\big|_0^{8/3} = \frac{1}{6}\tan^{-1}(8/3)$.

7.6.83 $\int \frac{3x+2}{x^2+4}\,dx$

Solution: Write $\int \frac{3x+2}{x^2+4}\,dx = 3\int \frac{x}{x^2+4}\,dx + 2\int \frac{1}{x^2+4}\,dx$. For the first integral, take $u = x^2+4$, and it becomes $\frac{3}{2}\int \frac{du}{u} = \frac{3}{2}\ln|x^2+4|$. For the second integral, write it as $\frac{1}{2}\int \frac{1}{(x/2)^2+1}\,dx$ and taking $u = x/2$ yields $\int \frac{1}{1+u^2}\,du = \tan^{-1}(x/2)$. Adding results in $\frac{3}{2}\ln|x^2+4| + \tan^{-1}(x/2) + C$.

7R.107 $\lim_{x\to 0} \frac{e^x}{e^x-1} - \frac{1}{x}$

Solution: This is indeterminate of type $\infty - \infty$. Writing the limit as a single fraction, we wish to compute $\lim_{x\to 0} \frac{xe^x - e^x + 1}{xe^x - x}$. Applying L'Hopital's rule, we get $\lim_{x\to 0} \frac{xe^x}{(x+1)e^x - 1}$. This is still $\frac{0}{0}$, so applying once more we get $\lim_{x\to 0} \frac{(x+1)e^x}{(x+2)e^x} = \frac{1}{2}$.

8.1.36 $\int x^3 e^{x^2}\,dx$

Solution: Set $t = x^2$, so $dt = 2x\,dx$. Subbing, the integral becomes $\frac{1}{2}\int te^t\,dt$. Integrate by parts with $u = t$ and $dv = e^t$ to get $\frac{1}{2}te^t - \frac{1}{2}e^t + C = \frac{1}{2}x^2e^{x^2} - \frac{1}{2}e^{x^2} + C$.

8.1.44 $\int \sin(\sqrt{x})\,dx$

Solution: Set $t = \sqrt{x}$ so $dt = \frac{1}{2\sqrt{x}}\,dx$. This says $dx = 2t\,dt$, so the integral becomes $2\int t\sin(t)\,dt$. Integrate by parts with $u = t$ and $dv = \sin(t)$ to get $2\int t\sin(t)\,dt = -2t\cos(t) + 2\sin(t) + C = -2\sqrt{x}\cos(\sqrt{x}) + 2\sin(\sqrt{x}) + C$.

8.1.82 $\int \frac{\ln(x)^2}{x^2}\,dx$

Solution: Set $t = \ln(x)$ so that $dt = \frac{1}{x}\,dx$ says $dx = e^t\,dt$. The integral transforms into $\int \frac{t^2}{e^{2t}} e^t\,dt = \int t^2 e^{-t}\,dt$. Integrate by parts with $u = t^2$ and $dv = e^{-t}$ to get $\int t^2 e^{-t}\,dt = -t^2 e^{-t} + \int 2te^{-t}\,dt$. Integrate by parts a second time to get $\int 2te^{-t}\,dt = -2te^{-t} - 2e^{-t}$. This yields $\int t^2 e^{-t}\,dt = -t^2 e^{-t} - 2te^{-t} - 2e^{-t} + C = -\ln(x)^2/x - 2\ln(x)/x - 2/x + C$.

1.

(a) Explain why L'Hopital's rule does not apply to $\lim_{x\to 0} \frac{\frac{1}{x} + xe^x}{e^x - 1}$. Compute the limit using an alternate method.

(b) Explain why L'Hopital's rule does not apply to $\lim_{x\to\infty} \frac{x}{x+\sin(x)}$. Compute the limit using an alternate method.

(c) Consider the expression $\lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}}$. Try to evaluate this limit using L'Hopital's rule. What goes wrong? Compute the limit using an alternate method.

Solution:

(a) Plugging in 0 gives $\frac{\infty}{0}$ which is not even an indeterminate form, so we definitely can't use L'Hopital's rule. This expression is just ∞, so the limit is infinite.

(b) We can't apply L'Hopital's rule because $\lim_{x\to\infty} \frac{f'(x)}{g'(x)} = \lim_{x\to\infty} \frac{1}{1+\cos(x)}$ does not exist! To compute the limit, divide the numerator and denominator by x: $\lim_{x\to\infty} \frac{x}{x+\sin(x)} = \lim_{x\to\infty} \frac{1}{1+\sin(x)/x} = 1$ since $\sin(x)/x \to 0$ as $x \to \infty$ (by squeezing between $-1/x$ and $1/x$, for example.)

(c) If you apply L'Hopital's rule, you get $\lim_{x\to\infty} \frac{1}{x(x^2+1)^{-1/2}} = \lim_{x\to\infty} \frac{\sqrt{x^2+1}}{x}$. If you apply it again, you'll get $\lim_{x\to\infty} \frac{x(x^2+1)^{-1/2}}{1} = \lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}}$, which is what we started with! Thus, we just get stuck in a loop.

To avoid this, there's two ways to compute the limit:

1. The usual 31A way. Divide numerator and denominator by x, so the limit becomes $\lim_{x\to\infty} \frac{1}{\sqrt{1+1/x^2}} = 1$.
2. If we take $L = \lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}}$, note that applying L'Hopital's rule yields $L = \frac{1}{L}$. This says $L^2 = 1$, and since the function is always positive, we must have $L = 1$.

Answers to even non-graded problems

7.5.38 : $\frac{1}{2}$

7.5.44 : 0

7.5.50 : $\frac{1}{e}$

7.6.82 : $\frac{1}{4+\ln(2)}e^{(4+\ln(2))x} + C$

7.6.108 : $\frac{1}{2}\sin^{-1}(t^2) + C$

7R.108 : $\frac{1}{6}$

8.1.18 : $\frac{4}{25}e^{3x}\sin(4x) + \frac{3}{25}e^{3x}\cos(4x) + C$

8.1.24 : $\frac{1}{2}x^2\ln(x)^2 - \frac{1}{2}x^2\ln(x) + \frac{1}{4}x^2 + C$