Selected Solutions to Homework 1

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7.1.29 Compute $f'(x)$ for $f(x) = \frac{e^{x+1}+x}{2e^x-1}$.

Solution: Use the quotient rule: $f'(x) = \frac{(e^{x+1}+1)(2e^x-1)-(e^{x+1}+x)(2e^x)}{(2e^x-1)^2}$.

7.1.76 Compute $\int e^x(e^{2x}+1)^3\,dx$.

Solution: Set $u = e^x$, so $du = e^x\,dx$. Plugging in, we're computing $\int (u^2+1)^3\,du$. Expanding out the integrand, $(u^2+1)^3 = u^6+3u^4+3u^2+1$, so we want $\int (u^6+3u^4+3u^2+1)\,du = \frac{1}{7}u^7+\frac{3}{5}u^5+u^3+u+C = \frac{1}{7}e^{7x}+\frac{3}{5}e^{5x}+e^{3x}+e^x+C$.

7.2.22 Show that $f(x) = (x^2+1)^{-1}$ is one-to-one on $(-\infty, 0]$ and find a formula for f^{-1} on this domain.

Solution: Note that $f'(x) = -2x(x^2+1)^{-2} > 0$ on $(-\infty, 0]$. This says that f is strictly increasing on it's domain, and therefore it's a one-to-one function. To find the inverse, set $y = (x^2+1)^{-1}$, so swapping gives $x = (y^2+1)^{-1}$. This gives $xy^2+x = 1$, so $y = \pm\sqrt{\frac{1-x}{x}}$. Since the domain of f is $(-\infty, 0]$ this means the range of f^{-1} is $(-\infty, 0]$, i.e. always negative. This means we take the negative branch of the square root, so $f^{-1}(x) = -\sqrt{\frac{1-x}{x}}$.

7.2.34 Find $g'(-1/2)$ for $g(x) = f^{-1}(x)$, $f(x) = \frac{x^3}{x^2+1}$.

Solution: The formula for the derivative of the inverse says $g'(-1/2) = \frac{1}{f'(g(-1/2))}$. By definition, $g(-1/2)$ is the number x such that $f(x) = -1/2$, i.e. the solution to $\frac{x^3}{x^2+1} = -\frac{1}{2}$. Doing some algebra, we want to solve $2x^3+x^2+1 = 0$, and by inspection $x = -1$ works. Therefore, $g(-1/2) = -1$, so $g'(-1/2) = \frac{1}{f'(-1)}$. By computation, $f'(x) = \frac{3x^2\cdot(x^2+1)-x^3\cdot(2x)}{(x^2+1)^2}$ so $f'(-1) = 1$, which yields $g'(-1/2) = 1$.

7.3.66 Compute $f'(x)$ for $f(x) = \frac{x(x+1)^3}{(3x-1)^2}$.

Solution: Taking logs, $\ln(f) = \ln(x)+3\ln(x+1)-2\ln(3x-1)$. Taking a derivative, $\frac{f'}{f} = \frac{1}{x}+\frac{3}{x+1}-\frac{6}{3x-1}$, so $f'(x) = \frac{x(x+1)^3}{(3x-1)^2}(\frac{1}{x}+\frac{3}{x+1}-\frac{6}{3x-1})$.

7.3.98 Compute $\int \frac{\ln(\ln(x))}{x\ln(x)}\,dx$.

Solution: Set $u = \ln(x)$, so $du = \frac{1}{x}\,dx$. The integral transforms to $\int \frac{\ln(u)}{u}\,du$. Set $v = \ln(u)$, so $dv = \frac{1}{u}\,du$. The integral becomes $\int v\,dv = \frac{1}{2}v^2 + C = \frac{1}{2}\ln(u)^2 + C = \frac{1}{2}\ln(\ln(x))^2 + C$.

1. Which is larger, π^e or e^π? In this problem, you'll use calculus to figure this out *without* a calculator.

(a) Let $f(x) = x^{1/x}$. What is the largest domain on which $f(x)$ can be defined? Show using calculus that $f(x)$ has a unique maximum value on this domain, and find the x-value where it happens. What is the range of $f(x)$?

(b) Which is larger: $f(\pi)$ or $f(e)$? Use this to determine which of π^e and e^π is larger. (*Hint: find a common exponent to raise both $f(\pi)$ and $f(e)$ to.*)

Solution:

(a) The domain of $f(x)$ is $x > 0$ (this is because exponents of negative numbers are not really well defined). To show that $f(x)$ has a unique maximum value on $(0, \infty)$, we'll do a standard maximization exercise. To compute $f'(x)$, first take a logarithm: $\ln(f) = \frac{\ln(x)}{x}$ so $\frac{f'}{f} = \frac{1-\ln(x)}{x^2}$, so $f'(x) = x^{1/x}(\frac{1-\ln(x)}{x^2})$. Setting $f'(x) = 0$, we need $1 - \ln(x) = 0$, so $x = e$. One sees that $f'(x) > 0$ for $0 < x < e$ and $f'(x) < 0$ for $e < x < \infty$, i.e. $x = e$ is a global maximum of $f(x)$. At $x = e$, $f(e) = e^{1/e}$ so the range of $f(x)$ is $(0, e^{1/e}]$.

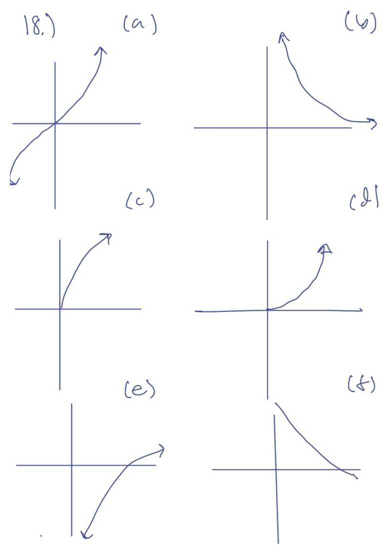
(b) Since $e < \pi$, this means $f(e) > f(\pi)$. Raising both sides to the πe power says $e^\pi = (e^{1/e})^{\pi e} > \pi^e = (\pi^{1/\pi})^{\pi e}$.

Answers to even non-graded problems

7.1.26 : $-\sin(te^{-2t}) \cdot (e^{-2t} - 2te^{-2t})$

7.1.30 : $e^{e^x} \cdot e^x$

7.2.18 :



7.3.32 : $\frac{1}{t} + \ln(5)$

7.3.72 : $x^{\cos(x)} \cdot (-\sin(x)\ln(x) + \frac{\cos(x)}{x})$

2. (a): $f^{(n)}(x) = (x+n)e^x$ (b): $f^{(n)}(x) = \frac{(-1)^{n+1}(n-1)!}{(1+x)^n}$ for $n \geq 1$.