Math 31B
Integration and Infinite Series

Practice Midterm

Directions: Do the problems below. You have 50 minutes to complete this exam. You may use a basic calculator without graphing or symbolic calculus capabilities. Show all your work. Write full sentences when necessary.

Name: ____________________________

UID: ______________________________

Question	Points	Score
1	10	
2	12	
3	12	
4	8	
Total:	42	

Formula Sheet

Trig Identities

- $\sin^2(x) + \cos^2(x) = 1$
- $\tan^2(x) + 1 = \sec^2(x)$
- $\sin(2x) = 2\sin(x)\cos(x)$
- $\cos(2x) = \cos^2(x) - \sin^2(x) = 2\cos^2(x) - 1 = 1 - 2\sin^2(x)$
- $\sin^2(x) = \frac{1-\cos(2x)}{2}$
- $\cos^2(x) = \frac{1+\cos(2x)}{2}$

Derivatives

- $\frac{d}{dx}(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$
- $\frac{d}{dx}b^x = b^x \ln(b)$
- $\frac{d}{dx}\log_b(x) = \frac{1}{x\ln(b)}$
- $\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$
- $\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2}$
- $\frac{d}{dx}\sec^{-1}(x) = \frac{1}{|x|\sqrt{x^2-1}}$

Integrals

- $\int u\,dv = uv - \int v\,du$
- $\int \frac{1}{x}\,dx = \ln|x| + C$
- $\int \tan(x)\,dx = \ln|\sec(x)| + C$
- $\int \cot(x)\,dx = \ln|\sin(x)| + C$
- $\int \sec(x)\,dx = \ln|\sec(x) + \tan(x)| + C$
- $\int \csc(x)\,dx = \ln|\csc(x) - \cot(x)| + C$

1. (10 pts.) Compute the following:

   (a) (5 pts.) $\frac{d}{dx}(2^x+1)^{\ln(2^x+1)}$

   Solution: Set $f = (2^x+1)^{\ln(2^x+1)}$. Then $\ln(f) = \ln(2^x+1)^2$, so taking a derivative gives $\frac{f'}{f} = \frac{2\ln(2^x+1)}{2^x+1}\cdot 2^x\ln(2)$. Solving for f' gives $f'(x) = (2^x+1)^{\ln(2^x+1)}(\frac{2\ln(2^x+1)}{2^x+1}\cdot 2^x\ln(2))$.

   (b) (5 pts.) $\lim_{x\to 0^+} xe^{1/x^2}$

   Solution: Rewrite the limit as $\lim_{x\to 0^+}\frac{e^{1/x^2}}{1/x}$, which is $\frac{\infty}{\infty}$. Using L'Hopital's this is $\lim_{x\to 0^+}\frac{e^{1/x^2}\cdot -2/x^3}{-1/x^2} = \lim_{x\to 0^+}\frac{2e^{1/x^2}}{x} = \frac{\infty}{0} = \infty$.

2. (12 pts.) Compute the following:

(a) (4 pts.) $g'(\pi/4)$, where $g(x) = f^{-1}(x)$ and $f(x) = \ln(x) + \tan^{-1}(x)$.

Solution: Using the formula, $g'(\pi/4) = 1/f'(g(\pi/4))$. Setting $g(\pi/4) = x$, this means $f(x) = \pi/4$. By inspection, $f(1) = \pi/4$, so $g(\pi/4) = 1$. This gives $g'(\pi/4) = 1/f'(1)$. Since $f'(x) = \frac{1}{x} + \frac{1}{1+x^2}$, $g'(\pi/4) = \frac{2}{3}$.

(b) (8 pts.) $\displaystyle\int \frac{e^x}{(1+e^{2x})^{3/2}}\,dx$

Solution: Set $u = e^x$, so $du = e^x\,dx$. The integral becomes $\int \frac{1}{(1+u^2)^{3/2}}\,du$. Set $u = \tan\theta$, so $du = \sec^2\theta\,d\theta$. After doing the substitution, the integral becomes $\int \frac{1}{\sec^3\theta}\sec^2\theta\,d\theta = \int \frac{1}{\sec\theta}\,d\theta = \int \cos\theta\,d\theta = \sin\theta + C$. Since $u = \tan\theta$, this means $\theta = \tan^{-1}(u)$. Drawing the appropriate right triangle yields $\sin\theta = \frac{u}{\sqrt{1+u^2}}$, and switching back to x-land results in $\frac{e^x}{\sqrt{1+e^{2x}}} + C$.

3. (12 pts.) Compute the following integrals:

(a) (6 pts.) $\int \frac{1}{2} \tan(\sqrt{x}) \sec(\sqrt{x})\, dx$

Solution: Set $t = \sqrt{x}$, so $dt = \frac{1}{2\sqrt{x}}\, dx$ gives $dx = 2t\, dt$. Pluging in, we get $\int t \tan(t) \sec(t)\, dt$. Integrate by parts with $u = t$ and $dv = \tan(t)\sec(t)$ and to get $t\sec(t) - \int \sec(t)\, dt = t\sec(t) - \ln|\sec(t) + \tan(t)| + C = \sqrt{x}\sec(\sqrt{x}) - \ln|\sec(\sqrt{x}) + \tan(\sqrt{x})| + C$.

(b) (6 pts.) $\int_0^{\pi/3} \tan^5(x)\sec^5(x)\, dx$

Solution: Rip out $\tan(x)\sec(x)$ to get $\int_0^{\pi/3} \tan(x)\sec(x)\tan^4(x)\sec^4(x)\, dx = \int_0^{\pi/3} \tan(x)\sec(x)(\sec^2(x) - 1)^2 \sec^4(x)\, dx$. Set $u = \sec(x)$, so this becomes $\int_1^2 (u^2 - 1)^2 u^4\, du = \int_1^2 (u^4 - 2u^2 + 1)u^4\, du = (\frac{1}{9}u^9 - \frac{2}{7}u^7 + \frac{1}{5}u^5)|_1^2 = \frac{8408}{315}$.

4. (8 pts.) Compute the following integral:

$$\int \frac{2x^3 + 2x^2 - 2x + 1}{x^2(x-1)^2}\,dx$$

Solution: Write $\frac{2x^3+2x^2-2x+1}{x^2(x-1)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{D}{(x-1)^2}$. Clearing denominators, $2x^3 + 2x^2 - 2x + 1 = Ax(x-1)^2 + B(x-1)^2 + Cx^2(x-1) + Dx^2$. Plug in $x = 0$ and you get $B = 1$. Plug in $x = 1$ and you get $D = 3$. Expanding out the right hand side and collecting like terms, we get $2x^3 + 2x^2 - 2x + 1 = (A+C)x^3 + (4-2A-C)x^2 + (A-2)x + 1$. Comparing coefficients, $A - 2 = -2$ and $A + C = 2$, so $A = 0$ and $C = 2$. Thus, the partial fraction decomposition is $\frac{2x^3+2x^2-2x+1}{x^2(x-1)^2} = \frac{1}{x^2} + \frac{2}{x-1} + \frac{3}{(x-1)^2}$. Integrating gives $-\frac{1}{x} + 2\ln|x-1| - \frac{3}{x-1} + C$.

(Challenge) Compute the following integral:

$$\int \sin^{-1}(\sqrt{x})\,dx$$

Note: this problem is too hard for a midterm, but it's great practice!

Solution: Set $t = \sqrt{x}$ so $2t\,dt = dx$. The integral becomes $\int 2t\sin^{-1}(t)\,dt$. Integrate by parts with $dv = 2t$ and $u = \sin^{-1}(t)$ to get $t^2\sin^{-1}(t) - \int \frac{t^2}{\sqrt{1-t^2}}\,dt$. For this latter integral, set $t = \sin(\theta)$ so $dt = \cos(\theta)\,d\theta$. The integral becomes $\int \frac{\sin^2(\theta)}{\cos(\theta)}\cos(\theta)\,d\theta = \int \sin^2(\theta)\,d\theta$. Use the power reduction formula to get $\int \sin^2(\theta)\,d\theta = \frac{\theta}{2} - \frac{1}{4}\sin(2\theta) + C$. Use $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$ and draw the appropriate right triangle to get $\cos(\theta) = \sqrt{1-t^2}$ so the integral in terms of t becomes $\frac{1}{2}\sin^{-1}(t) - \frac{t}{2}\sqrt{1-t^2} + C$. Combining with the previous result, $\int 2t\sin^{-1}(t)\,dt = t^2\sin^{-1}(t) - \frac{1}{2}\sin^{-1}(t) + \frac{t}{2}\sqrt{1-t^2} + C$. Subbing back in for x, we find $\int \sin(\sqrt{x})\,dx = x\sin^{-1}(x) - \frac{1}{2}\sin^{-1}(\sqrt{x}) + \frac{\sqrt{x}}{2}\sqrt{1-x} + C$.