Math 31B
Integration and Infinite Series

Final Exam

Directions: Do the problems below. You have 180 minutes to complete this exam. You may use a basic calculator without graphing or symbolic calculus capabilities. Show all your work. Write full sentences when necessary. If you need more space for scratch work, use the extra pages provided. **DO NOT WRITE ON THE BACK OF THE PAGE**.

Name: ________________________

UID: ________________________

Question	Points	Score
1	9	
2	8	
3	11	
4	10	
5	8	
6	13	
7	10	
8	8	
9	12	
10	11	
Total:	100	

Formula Sheet

Trig Identities

- $\sin^2(x) + \cos^2(x) = 1$
- $\tan^2(x) + 1 = \sec^2(x)$
- $\sin(2x) = 2\sin(x)\cos(x)$
- $\cos(2x) = \cos^2(x) - \sin^2(x) = 2\cos^2(x) - 1 = 1 - 2\sin^2(x)$
- $\sin^2(x) = \frac{1-\cos(2x)}{2}$
- $\cos^2(x) = \frac{1+\cos(2x)}{2}$

Derivatives

- $\frac{d}{dx} b^x = b^x \ln(b)$
- $\frac{d}{dx} \sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$
- $\frac{d}{dx} \tan^{-1}(x) = \frac{1}{1+x^2}$
- $\frac{d}{dx} \sec^{-1}(x) = \frac{1}{|x|\sqrt{x^2-1}}$

Integrals

- $\int u\,dv = uv - \int v\,du$
- $\int \frac{1}{x}\,dx = \ln|x| + C$
- $\int \tan(x)\,dx = \ln|\sec(x)| + C$
- $\int \sec(x)\,dx = \ln|\sec(x) + \tan(x)| + C$

Numerical Integration

- $M_N = \Delta x(f(c_1) + f(c_2) + \ldots + f(c_N))$, c_i midpoint of $[x_{i-1}, x_i]$.
- $T_N = \frac{1}{2}\Delta x(y_0 + 2y_1 + 2y_2 + \ldots + 2y_{N-1} + y_N)$, $y_i = f(x_i)$.
- $S_N = \frac{1}{3}\Delta x(y_0 + 4y_1 + 2y_2 + \ldots + 4y_{N-3} + 2y_{N-2} + 4y_{N-1} + y_N)$, $y_i = f(x_i)$.
- $\text{Error}(M_N) \leq \frac{K_2(b-a)^3}{24N^2}$
- $\text{Error}(T_N) \leq \frac{K_2(b-a)^3}{12N^2}$
- $\text{Error}(S_N) \leq \frac{K_4(b-a)^5}{180N^4}$

K_2 and K_4 are upper bounds of $|f''(x)|$ and $|f^{(4)}(x)|$ on the interval $[a, b]$ respectively.

Infinite Series

- $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for $x \in (-1, 1)$
- $e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$ for all $x \in \mathbb{R}$
- $\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$ for all $x \in \mathbb{R}$
- Taylor expansion of $f(x)$ centered at c: $\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$
- $|f(x) - T_N(x)| = |R_N(x)| \leq \frac{K_{N+1}}{(N+1)!}|x-c|^{N+1}$
- $|S - S_N| \leq a_{N+1}$ for $S = \sum_{n=0}^{\infty} (-1)^n a_n$
- $|S - S_N| \leq \int_N^{\infty} f(x)\,dx$ for $S = \sum_{n=0}^{\infty} f(n)$

K_{N+1} is an upper bound of $|f^{(N+1)}(z)|$ on the interval between x and c.

1. (9 pts.) True/False. Give short justifications for all your answers to receive full credit, including a counterexample if the statement is false.

   (a) (3 pts.) True/False: If $\sum_{n=0}^{\infty} a_n$ converges then $\sum_{n=0}^{\infty}(-1)^n a_n$ converges.

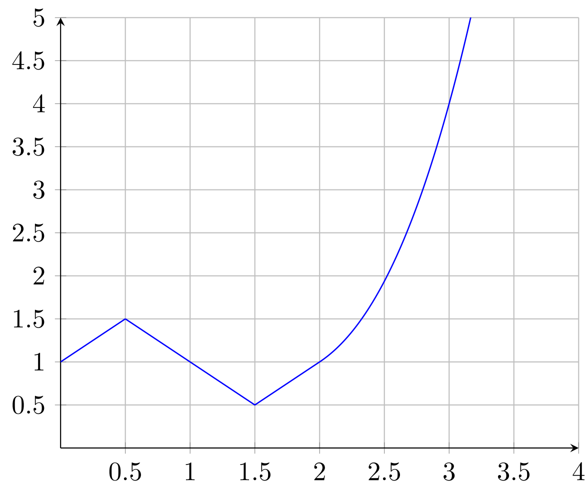
   Solution: False; For example, take $a_n = \frac{(-1)^n}{n+1}$. Then $\sum_{n=0}^{\infty} a_n$ is a convergent alternating series, but $\sum_{n=0}^{\infty}(-1)^n a_n$ diverges by comparison with the harmonic series.

   (b) (3 pts.) True/False: If a_n is the n-th partial sum of $\sum_{n=0}^{\infty} b_n$ and $\lim_{n\to\infty} a_n = 1$, then $\sum_{n=0}^{\infty} b_n$ diverges.

   Solution: False; by definition, this would mean that $\sum_{n=0}^{\infty} b_n = 1$, and therefore converges.

   (c) (3 pts.) True/False: The trapezoidal estimate T_3 overestimates $\int_0^3 f(x)\,dx$ for the function $f(x)$ below.

   Solution: True as seen from the drawing.



2. (8 pts.) Short answer. Answer the following questions. Justify your answers unless otherwise stated.

(a) (3 pts.) If $\lim_{n\to\infty} a_n = 2$, what is the interval of convergence of $\sum_{n=0}^{\infty} a_n x^n$?

Solution: The condition means $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = 1$ so by the ratio test, $R = 1$. At both $x = 1$ and $x = -1$ the general term does not tend to 0 and so the series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty}(-1)^n a_n$ diverge by the divergence test. Therefore, the interval of convergence is $(-1, 1)$.

(b) (2 pts.) What is the interval of convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n}$?

Solution: $[-1, 1)$. At $x = -1$ the convergence follows from the alternating series test, and $x = 1$ yields the harmonic series.

(c) (3 pts.) Give an example of a power series with interval of convergence $[7, 11)$.

Solution: The information provided says the power series must be centered at 9 and have radius of convergence 2. Modifying the series in the previous problem, $\sum_{n=0}^{\infty} \frac{1}{n2^n}(x-9)^n$ works.

3. (11 pts.)

(a) (8 pts.) Let

$$f(x) = \sum_{n=0}^{\infty} \frac{\tan^{-1}(n)}{1+n^2}(x-1)^{2n}$$

Determine the interval of convergence of $f(x)$. Justify the behavior at the endpoints carefully.

Solution: Running the ratio test, we need $\lim_{n\to\infty} \frac{\arctan(n+1)}{1+(n+1)^2} \cdot \frac{1+n^2}{\arctan(n)}|x-1|^2 < 1$. We have $\lim_{n\to\infty} \frac{\arctan(n+1)}{1+(n+1)^2} \cdot \frac{1+n^2}{\arctan(n)} = (\lim_{n\to\infty} \frac{\arctan(n+1)}{\arctan(n)}) \cdot (\lim_{n\to\infty} \frac{1+n^2}{1+(n+1)^2}) = 1$ since clearly $\lim_{n\to\infty} \frac{1+n^2}{1+(n+1)^2} = 1$ and $\lim_{n\to\infty} \frac{\arctan(n+1)}{\arctan(n)} = 1$ since both numerator and denominator tend to $\pi/2$. This says we need $|x-1|^2 < 1$, so $R = 1$ and we converge inside $(0, 2)$. At both endpoints the series is $\sum_{n=0}^{\infty} \frac{\arctan(n)}{1+n^2}$, and $\sum_{n=1}^{\infty} \frac{\arctan(n)}{1+n^2} \leq \sum_{n=1}^{\infty} \frac{\pi/2}{n^2} < \infty$, so the sum converges by direct comparison. This says the interval of convergence is $[0, 2]$.

Alternatively, one could do the integral test on $\sum_{n=0}^{\infty} \frac{\tan^{-1}(n)}{1+n^2}$ and directly compute $\int_0^{\infty} \frac{\tan^{-1}(x)}{1+x^2}\,dx$ to see it converges.

(b) (3 pts.) Compute the exact values of $f^{(2023)}(1)$ and $f^{(2024)}(1)$.

Solution: Compare coefficients with the general formula for the Taylor series. We have $\frac{f^{(2023)}(1)}{2023!} = 0$ since there is no $(x-1)^{2023}$ term in the sum, so $f^{(2023)}(1) = 0$. Similarly, $\frac{f^{(2024)}(1)}{2024!} = \frac{\arctan(1012)}{1+1012^2}$, so $f^{(2024)}(1) = \frac{2024!\arctan(1012)}{1+1012^2}$

4. (10 pts.) By starting with the list of known Maclaurin series on the formula sheet and performing the appropriate operations, find the Maclaurin series of

$$f(x) = \int_0^x \frac{t^3 - \ln(1+t^3)}{t^2}\,dt$$

and state the radius of convergence. To receive full credit, you must write your answer using summation notation. Make sure you show all your work!

Solution: Start with $\frac{1}{1-t} = \sum_{n=0}^{\infty} t^n$. Replacing t with $-t$ gives $\frac{1}{1+t} = \sum_{n=0}^{\infty}(-1)^n t^n$, so integrating results in $\ln(1+t) = \sum_{n=0}^{\infty} \frac{(-1)^n t^{n+1}}{n+1}$. Replacing t with t^3 gives $\ln(1+t^3) = \sum_{n=0}^{\infty} \frac{(-1)^n t^{3n+3}}{n+1}$. Subtracting and dividing by t^2 gives $\frac{t^3 - \ln(1+t^3)}{t^2} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} t^{3n+1}}{n+1}$, and integrating gives $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{3n+2}}{(n+1)(3n+2)}$. The radius of convergence is 1, as none of the operations we did at any step changed the radius of convergence (because $|t|^3 < 1$ if and only if $|t| < 1$, so the substitution didn't change the radius of convergence).

5. (8 pts.) Approximate $\displaystyle\int_0^{1/4} \tan^{-1}(4x^2)\,dx$ to within an error of $\frac{1}{10^3}$. You may use either infinite series or numerical integration, but make sure you carefully justify why your approximation has the correct level of accuracy, and show all your work. You do not need to fully simplify your end approximation.

Solution: The midpoint rule is probably the fastest way to proceed. With $f(x) = \arctan(4x^2)$, it's easy to compute $f''(x) = \frac{8-384x^4}{(1+16x^4)^2}$. This function is quite clearly decreasing on $[0, 1/4]$ as the numerator only gets smaller and the denominator only gets bigger as x gets larger. This has maximum value of 8 at $x = 0$, and so we can pick $K_2 = 8$. We then want to find N such that $\frac{8}{24N^2}(\frac{1}{4})^3 \leq \frac{1}{10^3}$, and by inspection $N = 3$ works. We then take 3 sub-intervals for our approximation, and so from the formula for the midpoint rule, our approximation is $\frac{1}{12}(f(1/24) + f(1/8) + f(5/24))$.

Alternatively, write $\tan^{-1}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$ (which can be found by integrating $\frac{1}{1+x^2}$, if you don't have this memorized), so $\tan^{-1}(4x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n 4^{2n+1} x^{4n+2}}{2n+1}$. Integrating then yields $\int_0^{1/4} \tan^{-1}(4x^2)\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n 4^{2n+1}}{(2n+1)(4n+3)4^{4n+3}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(4n+3)4^{2n+2}}$. By the alternating series error bound, we want to find N such that $\frac{1}{(2N+3)(4N+7)(4^{2N+4})} \leq \frac{1}{10^3}$. By inspection $N = 0$ works, so the first term of the sum is good enough and the approximation is $\frac{1}{3 \cdot 4^2} = \frac{1}{48}$.

6. (13 pts.)

(a) (5 pts.) Compute $\lim_{x\to 1^+} \sqrt{x-1}\ln(\ln(x))$

Solution: This is a $0\cdot\infty$ type limit so write this as $\lim_{x\to 1^+} \frac{\ln(\ln(x))}{(x-1)^{-1/2}}$ to make this an $\frac{\infty}{\infty}$ type limit. Doing L'Hopital once gives $\lim_{x\to 1^+} \frac{\frac{1}{\ln(x)}\cdot\frac{1}{x}}{-\frac{1}{2}(x-1)^{-3/2}} = \lim_{x\to 1^+} \frac{-2(x-1)^{3/2}}{x\ln(x)}$. Doing it a second time gives $\lim_{x\to 1^+} \frac{-3(x-1)^{1/2}}{\ln(x)+1} = 0$.

(b) (5 pts.) Use Taylor series to compute $\lim_{x\to 0} \dfrac{\cos(x^2)-1+x^4/2}{x^2(x-\sin(x))^2}$

Solution: We have $\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$ by taking a derivative of $\sin(x)$, so $\cos(x^2) = 1 - x^4/2 + x^8/24 - \ldots$. Therefore, we want to compute $\lim_{x\to 0} \frac{\frac{1}{24}x^8+\ldots}{x^2(\frac{1}{6}x^3+\ldots)^2} = \lim_{x\to 0} \frac{\frac{1}{24}x^8+\ldots}{(\frac{1}{6}x^4+\ldots)^2} = \lim_{x\to 0} \frac{\frac{1}{24}+\ldots}{(\frac{1}{6}+\ldots)^2} = \frac{3}{2}$ since all the terms hidden behind the $\ldots$ have a power of x larger than 1, and so disappear when plugging in $x=0$.

(c) (3 pts.) Order the following functions from slowest growing to fastest growing as $x\to\infty$. Give a short justification as to how you know your ordering is correct.

$$x^2\ln(x), x\ln(x)^2, x^{5/4}, \ln(\ln(x)), \ln(x)^4$$

Solution: The order is $\ln(\ln(x)) \ll \ln(x)^4 \ll x\ln(x)^2 \ll x^{5/4} \ll x^2\ln(x)$, where $\ll$ means "eventually less than". $\ln(x)$ eventually grows slower than any power of x, and so this also means $\ln(\ln(x))$ eventually grows slower than $\ln(x)$ by taking a logarithm. The first $\ll$ is then obvious, the second/third from the first statement, and the last because $\ln(x) > 1$ eventually so $x^{5/4} < x^2 < x^2\ln(x)$ eventually.

7. (10 pts.)

(a) (5 pts.) Compute $\displaystyle\int_0^1 \frac{120x^2}{(4-x^2)^{7/2}}\,dx$

Solution: Set $x = 2\sin\theta$. After substituting in and simplifying, the integral becomes $\int_0^{\pi/6} \frac{960\sin^2(\theta)}{128\cos^6(\theta)}\,d\theta = \frac{15}{2}\int_0^{\pi/6}\tan^2(\theta)\sec^4(\theta)\,d\theta$. Using $\sec^2(\theta) = \tan^2(\theta)+1$, this is $\frac{15}{2}\int_0^{\pi/6}\tan^2(\theta)(\tan^2(\theta)+1)\sec^2(\theta)\,d\theta$, which after a substitution of $u = \tan(\theta)$ is $\frac{15}{2}(\frac{\tan^5(\theta)}{5} + \frac{\tan^3(\theta)}{3}\Big|_0^{\pi/6}) = \frac{1}{\sqrt{3}}$.

(b) (5 pts.) Compute $\displaystyle\int \frac{x^3+2x^2+2}{x^2(x^2+2)}\,dx$

Solution: The general partial fraction decomposition looks like $\frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+2}$, so we want to solve $x^3 + 2x^2 + 2 = Ax(x^2+2) + B(x^2+2) + (Cx+D)x^2$. Expanding, this means $x^3+2x^2+2 = (A+C)x^3 + (B+C)x^2 + 2Ax + 2B$. Comparing coefficients yields $A = 0, B = C = D = 1$ and so we want to compute $\int (\frac{1}{x^2} + \frac{x+1}{x^2+2})\,dx = \int \frac{1}{x^2} + \frac{x}{x^2+2} + \frac{1}{x^2+2}\,dx = -\frac{1}{x} + \frac{1}{2}\ln(x^2+2) + \frac{1}{\sqrt{2}}\arctan(\frac{x}{\sqrt{2}}) + C$.

8. (8 pts.) Determine if $\displaystyle\int_0^\infty \frac{2^x}{\sqrt{x}(1+e^x)}\,dx$ converges or diverges. Justify your answer carefully.

Solution: We have $\int_0^\infty \frac{2^x}{\sqrt{x}(1+e^x)}\,dx = \int_0^1 \frac{2^x}{\sqrt{x}(1+e^x)}\,dx + \int_1^\infty \frac{2^x}{\sqrt{x}(1+e^x)}\,dx$. For the first integral, we have $\int_0^1 \frac{2^x}{\sqrt{x}(1+e^x)}\,dx \leq \int_0^1 \frac{2}{\sqrt{x}}\,dx = 4$, so it converges. For the second integral, we have $\int_1^\infty \frac{2^x}{\sqrt{x}(1+e^x)}\,dx \leq \int_1^\infty (\frac{2}{e})^x\,dx = -\frac{2/e}{\ln(2/e)}$ by direct computation. Since both pieces converge, the entire integral converges.

9. (12 pts.) Determine if the following infinite series converge (conditionally or absolutely, if applicable) or diverge. Justify your answers carefully.

   (a) (6 pts.) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 + \ln(n)}{6^n}$

   Solution: We have $\frac{n^2+\ln(n)}{6^n} \leq \frac{2\cdot 2^n}{6^n} = 2(\frac{1}{3})^n$, and $\sum_{n=1}^{\infty} 2(\frac{1}{3})^n$ is a convergent geometric series because $1/3 < 1$, so the series converges by a direct comparison. Alternatively, you can do the ratio test.

   (b) (6 pts.) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2 + 4^{-n}}$

   Solution: We have $\sum_{n=1}^{\infty} \frac{n}{n^2+4^{-n}} \approx \sum_{n=1}^{\infty} \frac{1}{n}$ which diverges. Running LCT with $\sum_{n=1}^{\infty} \frac{n}{n^2+4^{-n}}$ and $\sum_{n=1}^{\infty} \frac{1}{n}$, we see that $\lim_{n\to\infty} \frac{n/(n^2+4^{-n})}{1/n} = \lim_{n\to\infty} \frac{n^2}{n^2+4^{-n}} = 1$, so the series diverges.

   On the other hand, we have $\lim_{n\to\infty} \frac{n}{n^2+4^{-n}} = 0$ and the function $f(x) = \frac{x}{x^2+4^{-x}}$ is decreasing because $f'(x) = \frac{-x^2+4^{-x}-x\ln(4)4^{-x}}{(x^2+4^{-x})^2} < 0$ eventually since $-x^2$ is the dominant term. Therefore by the alternating series test, the alternating series converges, and so the convergence is conditional.

10. (11 pts.) Define a sequence a_n by $a_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} x^2\cos(nx)\,dx$.

(a) (8 pts.) Find an explicit formula for a_n for $n \geq 1$.

Solution: Integrate by parts with $u = x^2$ and $dv = \cos(nx)$ to get $\frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos(nx)\,dx = \frac{1}{\pi}(\frac{1}{n}x^2\sin(nx)|_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\frac{2}{n}x\sin(nx)\,dx) = -\frac{2}{n\pi}\int_{-\pi}^{\pi} x\sin(nx)\,dx$ since $\sin(\pi n) = 0$ for all integer n. Integrate by parts again to get $-\frac{2}{n\pi}\int_{-\pi}^{\pi} x\sin(nx)\,dx = \frac{2x}{n^2\pi}\cos(nx)|_{-\pi}^{\pi} + \frac{2}{n\pi}\int_{-\pi}^{\pi}\cos(nx)\,dx = \frac{2x}{n^2\pi}\cos(nx)|_{-\pi}^{\pi} = (-1)^n\frac{4}{n^2\pi}$ because $\cos(\pi n) = \cos(-\pi n) = (-1)^n$. Therefore, $a_n = (-1)^n\frac{4}{n^2}$.

(b) (3 pts.) For which values of x does $\displaystyle\sum_{n=1}^{\infty} a_n\cos(nx)$ converge?

Solution: The series is $\sum_{n=1}^{\infty}(-1)^n\frac{4}{n^2}\cos(nx)$. Taking an absolute value, we see $\sum_{n=1}^{\infty}|(-1)^n\frac{4}{n^2}\cos(nx)| \leq \sum_{n=1}^{\infty}\frac{4}{n^2} < \infty$ regardless of x, so the series converges (absolutely!) for all $x \in \mathbb{R}$.