Math 31B
Integration and Infinite Series

Final Exam

Directions: Do the problems below. You have 180 minutes to complete this exam. You may use a basic calculator without graphing or symbolic calculus capabilities. Show all your work. Write full sentences when necessary. If you need more space for scratch work, use the extra pages provided. **DO NOT WRITE ON THE BACK OF THE PAGE**.

Name: ________________________

UID: ________________________

Question	Points	Score
1	9	
2	8	
3	11	
4	10	
5	8	
6	13	
7	10	
8	8	
9	12	
10	11	
Total:	100	

Formula Sheet

Trig Identities

- $\sin^2(x) + \cos^2(x) = 1$
- $\tan^2(x) + 1 = \sec^2(x)$
- $\sin(2x) = 2\sin(x)\cos(x)$
- $\cos(2x) = \cos^2(x) - \sin^2(x) = 2\cos^2(x) - 1 = 1 - 2\sin^2(x)$
- $\sin^2(x) = \frac{1-\cos(2x)}{2}$
- $\cos^2(x) = \frac{1+\cos(2x)}{2}$

Derivatives

- $\frac{d}{dx} b^x = b^x \ln(b)$
- $\frac{d}{dx} \sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$
- $\frac{d}{dx} \tan^{-1}(x) = \frac{1}{1+x^2}$
- $\frac{d}{dx} \sec^{-1}(x) = \frac{1}{|x|\sqrt{x^2-1}}$

Integrals

- $\int u\,dv = uv - \int v\,du$
- $\int \frac{1}{x}\,dx = \ln|x| + C$
- $\int \tan(x)\,dx = \ln|\sec(x)| + C$
- $\int \sec(x)\,dx = \ln|\sec(x) + \tan(x)| + C$

Numerical Integration

- $M_N = \Delta x(f(c_1) + f(c_2) + \ldots + f(c_N))$, c_i midpoint of $[x_{i-1}, x_i]$.
- $T_N = \frac{1}{2}\Delta x(y_0 + 2y_1 + 2y_2 + \ldots + 2y_{N-1} + y_N)$, $y_i = f(x_i)$.
- $S_N = \frac{1}{3}\Delta x(y_0 + 4y_1 + 2y_2 + \ldots + 4y_{N-3} + 2y_{N-2} + 4y_{N-1} + y_N)$, $y_i = f(x_i)$.
- $\text{Error}(M_N) \leq \frac{K_2(b-a)^3}{24N^2}$
- $\text{Error}(T_N) \leq \frac{K_2(b-a)^3}{12N^2}$
- $\text{Error}(S_N) \leq \frac{K_4(b-a)^5}{180N^4}$

K_2 and K_4 are upper bounds of $|f''(x)|$ and $|f^{(4)}(x)|$ on the interval $[a, b]$ respectively.

Infinite Series

- $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for $x \in (-1, 1)$
- $e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$ for all $x \in \mathbb{R}$
- $\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$ for all $x \in \mathbb{R}$
- Taylor expansion of $f(x)$ centered at c: $\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$
- $|f(x) - T_N(x)| = |R_N(x)| \leq \frac{K_{N+1}}{(N+1)!}|x-c|^{N+1}$
- $|S - S_N| \leq a_{N+1}$ for $S = \sum_{n=0}^{\infty} (-1)^n a_n$
- $|S - S_N| \leq \int_N^{\infty} f(x)\,dx$ for $S = \sum_{n=0}^{\infty} f(n)$

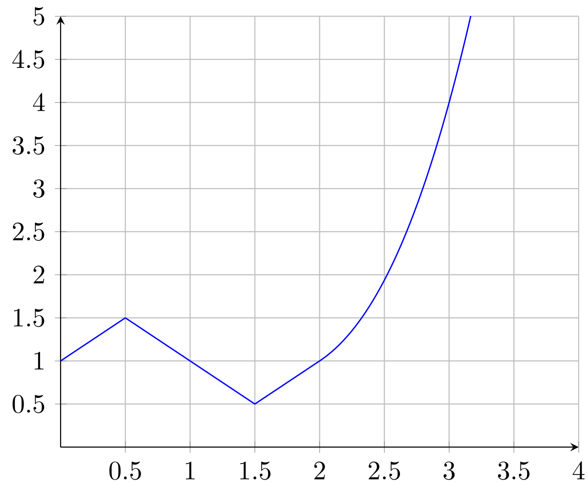
K_{N+1} is an upper bound of $|f^{(N+1)}(z)|$ on the interval between x and c.

1. (9 pts.) True/False. Give short justifications for all your answers to receive full credit, including a counterexample if the statement is false.

   (a) (3 pts.) True/False: If $\sum_{n=0}^{\infty} a_n$ converges then $\sum_{n=0}^{\infty} (-1)^n a_n$ converges.

   (b) (3 pts.) True/False: If a_n is the n-th partial sum of $\sum_{n=0}^{\infty} b_n$ and $\lim_{n\to\infty} a_n = 1$, then $\sum_{n=0}^{\infty} b_n$ diverges.

   (c) (3 pts.) True/False: The trapezoidal estimate T_3 overestimates $\int_0^3 f(x)\,dx$ for the function $f(x)$ below.



2. (8 pts.) Short answer. Answer the following questions. Justify your answers unless otherwise stated.

(a) (3 pts.) If $\lim_{n\to\infty} a_n = 2$, what is the interval of convergence of $\sum_{n=0}^{\infty} a_n x^n$?

(b) (2 pts.) What is the interval of convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n}$?

(c) (3 pts.) Give an example of a power series with interval of convergence $[7, 11)$.

3. (11 pts.)

(a) (8 pts.) Let

$$f(x) = \sum_{n=0}^{\infty} \frac{\tan^{-1}(n)}{1+n^2}(x-1)^{2n}$$

Determine the interval of convergence of $f(x)$. Justify the behavior at the endpoints carefully.

(b) (3 pts.) Compute the exact values of $f^{(2023)}(1)$ and $f^{(2024)}(1)$.

4. (10 pts.) By starting with the list of known Maclaurin series on the formula sheet and performing the appropriate operations, find the Maclaurin series of

$$f(x) = \int_0^x \frac{t^3 - \ln(1+t^3)}{t^2}\, dt$$

and state the radius of convergence. To receive full credit, you must write your answer using summation notation. Make sure you show all your work!

5. (8 pts.) Approximate $\displaystyle\int_0^{1/4} \tan^{-1}(4x^2)\,dx$ to within an error of $\frac{1}{10^3}$. You may use either infinite series or numerical integration, but make sure you carefully justify why your approximation has the correct level of accuracy, and show all your work. You do not need to fully simplify your end approximation.

6. (13 pts.)

(a) (5 pts.) Compute $\lim_{x \to 1^+} \sqrt{x-1}\ln(\ln(x))$

(b) (5 pts.) Use Taylor series to compute $\lim_{x \to 0} \frac{\cos(x^2) - 1 + x^4/2}{x^2(x - \sin(x))^2}$

(c) (3 pts.) Order the following functions from slowest growing to fastest growing as $x \to \infty$. Give a short justification as to how you know your ordering is correct.

$$x^2 \ln(x), x \ln(x)^2, x^{5/4}, \ln(\ln(x)), \ln(x)^4$$

7. (10 pts.)

(a) (5 pts.) Compute $\displaystyle\int_0^1 \frac{120x^2}{(4-x^2)^{7/2}}\,dx$

(b) (5 pts.) Compute $\displaystyle\int \frac{x^3+2x^2+2}{x^2(x^2+2)}\,dx$

8. (8 pts.) Determine if $\displaystyle\int_0^\infty \frac{2^x}{\sqrt{x}(1+e^x)}\,dx$ converges or diverges. Justify your answer carefully.

9. (12 pts.) Determine if the following infinite series converge (conditionally or absolutely, if applicable) or diverge. Justify your answers carefully.

   (a) (6 pts.) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 + \ln(n)}{6^n}$

   (b) (6 pts.) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2 + 4^{-n}}$

10. (11 pts.) Define a sequence a_n by $a_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} x^2 \cos(nx)\, dx$.

(a) (8 pts.) Find an explicit formula for a_n for $n \geq 1$.

(b) (3 pts.) For which values of x does $\displaystyle\sum_{n=1}^{\infty} a_n \cos(nx)$ converge?

11. Extra credit! You must show all your work for full credit. (Only work on these if you have time!)

    (a) (5 pts.) Compute $\lim_{x\to\infty} \frac{\ln(x)}{x} \int_0^x \frac{1}{\ln(t)}\, dt$

    (b) (5 pts.) Determine if $\int_0^1 \frac{\sqrt{\sin(x)}}{1-\cos(x)}\, dx$ converges or diverges.

Scratch Paper

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