Quiz 4 Solutions

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1. Define an equivalence relation $\sim$ on $X = \mathbb{R} \setminus \{0\}$ by $x \sim y \iff x$ and y have the same sign , i.e. are both positive or both negative. For each of the following functions, determine if they are well-defined or not.

(a) $f : X/\sim \to \{-1, 1\}$ defined by $f([x]) = \frac{x}{|x|}$.

(b) $g : X/\sim \to \mathbb{R}$ by $g([x]) = x^2$.

(c) $h : X/\sim \to X/\sim$ by $h([x]) = [-x^3]$.

Solution:

(a) Notice that

$$\frac{x}{|x|} = \begin{cases} +1 & x > 0 \\ -1 & x < 0 \end{cases}$$

simply returns the sign of x. Therefore, if $[x] = [y]$, then x and y are both either positive or negative, and so $\frac{x}{|x|} = \frac{y}{|y|}$ says f is well-defined.

(b) Not well-defined; $[1] = [2]$ but $g([1]) = 1 \neq 4 = g([2])$.

(c) If $x > 0$, then $-x^3 < 0$ and if $x < 0$, then $-x^3 > 0$, i.e. the map $x \to -x^3$ just flips the sign of x. Therefore if $[x] = [y]$, x and y have the same sign, so that $-x^3$ and $-y^3$ have the same sign, so $[-x^3] = [-y^3]$ says f is well-defined.

2. Find the remainder when 20^{100} is divided by 17.

Solution: We wish to compute $20^{100} \bmod 17 \equiv 3^{100} \bmod 17$. By Fermat's little theorem, $3^{16} \equiv 1 \bmod 17$, and since $100 = 6 \cdot 16 + 4$, $3^{100} \equiv 3^4 \equiv 13 \bmod 17$.