Quiz 2 Solutions

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1. Let $a, b, c \in \mathbb{Z}$ be such that $a \mid b$ and $b \mid c$. Prove that $a \mid c$.

Solution: Since $a \mid b$, we may write $b = ak$ for some $k \in \mathbb{Z}$. Since $b \mid c$, we may write $c = b\ell$ for some $\ell \in \mathbb{Z}$. Then $c = a(k\ell)$, and $k\ell \in \mathbb{Z}$, so that $a \mid c$.

2. Compute $\gcd(4784, 2231)$.

Solution: Run the Euclidean algorithm:

$$4784 = 2 \cdot 2231 + 322$$

$$2231 = 6 \cdot 322 + 299$$

$$322 = 1 \cdot 299 + 23$$

$$299 = 13 \cdot 23 + 0$$

This says $\gcd(4784, 2231) = 23$.

3. Let F_n be the Fibonacci sequence defined by $F_0 = 1$, $F_1 = 1$, and $F_{n+1} = F_n + F_{n-1}$ for $n \geq 2$. Show that $\gcd(F_n, F_{n-1}) = 1$ for all $n \geq 1$.

Solution: We prove this by induction. For $n = 1$, we have $\gcd(F_1, F_0) = \gcd(1, 1) = 1$. Suppose for some integer k that $\gcd(F_k, F_{k-1}) = 1$. It's clear that $\{F_n\}$ is an increasing sequence, so in particular we have $F_{k-1} < F_k$, so the first step in the Euclidean algorithm applied to the pair (F_{k+1}, F_k) says $F_{k+1} = F_k + F_{k-1}$, which then says the second step is the first step of the Euclidean algorithm for the pair (F_k, F_{k-1}). Since the Euclidean algorithm applied to a pair of integers terminates in their gcd, this then says $\gcd(F_{k+1}, F_k) = \gcd(F_k, F_{k-1})$, so that applying the induction hypothesis says $\gcd(F_{k+1}, F_k) = 1$. By induction, this says $\gcd(F_n, F_{n-1}) = 1$ for all $n \geq 1$.