Homework 6 Solutions

Tim Smits

**1.** Define $\sim$ on $\mathbb{R}$ by $x \sim y \iff x - y \in \mathbb{Z}$. Prove that $\sim$ is an equivalence relation.

**Solution:** Note that for any $x \in \mathbb{R}, x \sim x$ because $x - x = 0 \in \mathbb{Z}$. If $x \sim y$, then $y - x = -(x - y) \in \mathbb{Z}$ so that $y \sim x$. Finally, if $x \sim y$ and $y \sim z$, then $x - z = (x - y) + (y - z) \in \mathbb{Z}$, so that $x \sim z$.

**2.** What is $[.3]$? What is $[1]$? What is $[\pi]$?

**Solution:** $[.3] = \{.3 + k : k \in \mathbb{Z}\}$, $[1] = \mathbb{Z}$, and $[\pi] = \{\pi + k : k \in \mathbb{Z}\}$.

**3.** Give a complete set of representatives for the equivalence relation $\sim$.

**Solution:** The claim is that $[0, 1)$ is a set of representatives for $\sim$. Firstly, note that each class $[x]$ for $x \in [0, 1)$ is different, because no two real numbers in the interval $[0, 1)$ differ by an integer. Next, for $x \in \mathbb{R}$, write $c = x - \lfloor x \rfloor \in [0, 1)$. This says $x - c = \lfloor x \rfloor \in \mathbb{Z}$, so that $x \sim c$. This says for any $x \in \mathbb{R}$, that $[x]$ has a representative in $[0, 1)$ as desired.

**4.** Which of the following are well-defined functions on $\mathbb{R}/\sim$?

(a) $f([x]) = 2x$

(b) $g([x]) = [2x]$

(c) $h([x]) = [\frac{1}{2}x]$

(d) $u([x]) = [x^2]$

**Solution:**

(a) Not well-defined; $[1] = [2]$ but $2 \neq 4$.

(b) Well-defined; if $[x] = [y]$, then $y = x + k$ for some $k \in \mathbb{Z}$. Then $g([y]) = g([x + k]) = [2(x + k)] = [2x + 2k] = [2x]$ because $2k \in \mathbb{Z}$.

(c) Not well-defined; $[1] = [2]$ but $[\frac{1}{2}] \neq [1]$,

(d) Not well-defined; $[\frac{1}{3}] = [\frac{4}{3}]$, but $[\frac{1}{9}] \neq [\frac{16}{9}] = [\frac{7}{9}]$.

**5.** Define $f : \mathbb{R}/\sim \to \mathbb{R}^2$ by $f([x]) = (\cos(2\pi x), \sin(2\pi x))$. Show that $f$ is well-defined, injective, and find it's image.

**Solution:** Suppose that $[x] = [y]$, then $y = x + k$ for some $k \in \mathbb{Z}$. Then $f([y]) = f([x + k]) = (\cos(2\pi(x + k)), \sin(2\pi(x + k))) = (\cos(2\pi x), \sin(2\pi x)) = f([x])$ since cosine and sine are $2\pi$-periodic, so that $f$ is well-defined. Now suppose that $f([x]) = f([y])$, this says $\cos(2\pi x) = \cos(2\pi y)$ and $\sin(2\pi x) = \sin(2\pi y)$. This then says $2\pi y = 2\pi k + 2\pi x$ for some $k \in \mathbb{Z}$, or equivalently, $y = x + k$, so that $[x] = [y]$. This then says that $f$ is injective. Finally, it's clear that the image of $f$ is simply the unit circle.

**6.** Using the equivalence relation on the set $X = \mathbb{Z}^2 \backslash \{(0,0)\}$ defined by $(a, b) \sim (c, d) \iff ad = bc$, what is the equivalence class of $(3, 4)$?

**Solution:** The claim is that $[(3, 4)] = \{(3k, 4k) : k \neq 0 \in \mathbb{Z}\}$. Firstly, note that every element of this set is an element of $[(3, 4)]$ because $3 \cdot (4k) = 4 \cdot (3k) = 12k$, so that $(3, 4) \sim (3k, 4k)$ for any $k \neq 0$, so we just need to show that every ordered pair that relates to $(3, 4)$ is of this form. Suppose that $(3, 4) \sim (a, b)$ for some integers $a, b \neq 0$, so that $3b = 4a$. Since $3 \mid 4a$ and $\gcd(3, 4) = 1$, this says that $3 \mid a$. Similary, $4 \mid b$, so write $a = 3m$ and $b = 4n$ for some integers $m, n$. This says $12n = 12m$, so that $n = m$. Setting $n = m = k$, this says $(a, b) = (3k, 4k)$ for some integer $k$ as desired.

**7.** Is the function $g : X/\sim \to \mathbb{Q}$ defined by $g([(a, b)]) = ab$ well defined?

**Solution:** It is not well-defined; note that $(1, 2) \sim (2, 4)$ but $2 \neq 8$.

**8.** Define $f : X/\sim \to \mathbb{Q}$ by $f([(a, b)]) = \frac{a}{b}$. Prove that $f$ is a well-defined bijection.

**Solution:** Suppose that $(a, b) \sim (c, d)$, so that $ad = bc$. Then $f([(a, b)]) = \frac{a}{b} = \frac{ac}{bc} = \frac{ac}{ad} = \frac{c}{d} = f([(c, d)])$, so that $f$ is well-defined. The function $f$ is clearly surjective, because for any rational number $\frac{a}{b} \in \mathbb{Q}$, we have $f([(a, b)]) = \frac{a}{b}$. Now suppose that $f([(a, b)]) = f([(c, d)])$. This says $\frac{a}{b} = \frac{c}{d}$, so that $ad = bc$ gives $(a, b) \sim (c, d)$, i.e. $[(a, b)] = [(c, d)]$. Therefore $f$ is injective, and thus a bijection.