Primes à la Euler

Thm: (Euler) There are inf. many primes

Proof:

Suppose $p_1, \ldots, p_k$ are all the primes.

$$\frac{1}{1-\frac{1}{p_i}} = \sum_{n=0}^{\infty} \frac{1}{p_i^{\,n}}$$

Consider the product

$$\prod_{i=1}^{k} \frac{1}{1-\frac{1}{p_i}} = \left(\sum_{n=0}^{\infty} \frac{1}{p_1^{\,n}}\right) \cdots \left(\sum_{n=0}^{\infty} \frac{1}{p_k^{\,n}}\right)$$

$$\left(1+\frac{1}{p_1}+\frac{1}{p_1^2}+\cdots\right)\left(1+\frac{1}{p_2}+\cdots\right)\cdots\left(1+\frac{1}{p_k}+\cdots\right)$$

By FTA, for any integer, we can

write $m = p_1^{e_1} \cdots p_k^{e_k}$

By considering the appropriate terms in RHS,

$$\Rightarrow \quad RHS = \sum_{n=1}^{\infty} \frac{1}{n}.$$

Contradiction, b/c LHS is finite while $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges!

Euler's proof actually gives us more info about primes that Euclid's doesn't:

Thm: (Euler)

$$\sum_{p \text{ prime}} \frac{1}{p} \text{ diverges}$$

Proof:

For $N \geq 2$ we have

$$\prod_{p \leq N} \frac{1}{1-\frac{1}{p}} = \prod_{p \leq N} \left(1 + \frac{1}{p} + \frac{1}{p^2} + \cdots\right)$$

by unique factorization, we can get $1/m$ as a term in the RHS for any $m \leq N$

by picking correct exponents, where $m = p_1^{e_1} \cdots p_k^{e_k}$. Therefore,

$$\sum_{n \le N} \frac{1}{n} \le \prod_{p \le N} \frac{1}{1 - \frac{1}{p}}$$

Now take log:

$$\log\left(\sum_{n \le N} \frac{1}{n}\right) \le \sum_{p \le N} \log\left(\frac{1}{1 - \frac{1}{p}}\right)$$

$$= \sum_{p \le N} -\log\left(1 - \frac{1}{p}\right)$$

From Calculus, $-\log(1-x)$

$$\approx x + \frac{1}{2}x^2 + \cdots$$

valid for $|x| < 1$.

It's not good enough to just use an "approximation", but it turns out for

$0 \le x \le \frac{1}{2}$ that

$$-\log(1-x) \le x + x^2.$$

So

$$-\log\left(1-\frac{1}{p}\right) \le \frac{1}{p} + \frac{1}{p^2}$$

$\Longrightarrow$

$$\log\Big(\sum_{n\le N}\frac{1}{n}\Big) \le \sum_{p\le N}\frac{1}{p}+\frac{1}{p^2}$$

$$= \sum_{p\le N}\frac{1}{p} + \sum_{p\le N}\frac{1}{p^2}$$

As $N \to \infty$,

$$\sum_{p\le N}\frac{1}{p^2} \le \sum_{n=1}^{\infty}\frac{1}{n^2} < \infty$$

$$\Longrightarrow \quad \sum_{p}\frac{1}{p} \text{ diverges.}$$

Takeaway: able to get information about primes by using Calculus!

Rmk:

Easy, from integral test

$$\sum_{1 \le n \le x} \frac{1}{n} \approx \log(x)$$

$$\sum_{1 \le p \le x} \frac{1}{p} \approx \log(\log(x))$$

Merten's Second Thm, Harder!

Big theme in number theory:
Study behavior of
"counting" functions using
long term/average behavior
that calculus is suited
to deal with. E.g. study
primes w/ $\pi(x)$.

<u>Thm:</u> (Prime Number Thm)

$$\lim \pi(x) / x/\ln(x) = 1.$$

$x \to \infty$

i.e. $\pi(x) \sim x/\ln(x)$

PNT says is

primes $\leq x$ is roughly

$x/\ln(x)$, so

$$\frac{\pi(x)}{x} \underset{\sim}{\wedge} \frac{1}{\ln(x)} \quad \text{i.e.}$$

primes have a "density" of

$\frac{1}{\ln(x)}$. How else to approximate totals w/ density? Integrate!

Gauss came up w/ a better approximation:

$$\mathrm{Li}(x) = \int_2^x \frac{1}{\ln(t)}\,dt$$

Hueristically, $\mathrm{Li}(x)$ should approximate $\pi(x)$. In fact, $\mathrm{Li}(x)$ is a much <u>better</u> approximation to $\pi(x)$ than $\frac{x}{\ln(x)}$ is!

How good? This is the Riemann Hypothesis:

There is some constant C such that

$$|Li(x) - \pi(x)| \leq C\sqrt{x}\log(x)$$

for all x large enough.

i.e. the error between $Li(x)$ and $\pi(x)$ behaves like $\sqrt{x}\log(x)$.

A rough idea: we expect

$$\left(C < \frac{131}{984\pi}, \quad \text{valid for } x \geq 2657 \right)$$

How to actually estimate $Li(x)$? Use integration by parts!

$$Li(x) = \int_2^x \frac{1}{\ln(t)}\,dt$$

$$u = \frac{1}{\ln(t)} \qquad dv = 1$$

$$du = -\frac{1}{t\ln(t)^2} \qquad v = t$$

$$\mathrm{Li}(x) = \frac{t}{\ln(t)}\Bigg|_2^x + \int_2^x \frac{1}{\ln(t)^2}\,dt$$

Repeatedly do this:

$$u = \frac{1}{\ln(t)^2} \qquad dv = 1$$

etc.

$$\mathrm{Li}(x) \approx \frac{x}{\ln(x)} + \frac{x}{\ln(x)^2} + \frac{2x}{\ln(x)^3} + \dots$$

Some data:

x	$\pi(x)$	$Li(x)$
10^4	1229	≈ 1229
10^5	9592	≈ 9571
10^6	78498	≈ 78380
10^{10}	455052511	≈ 454793911