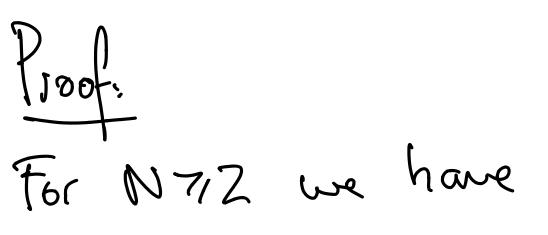
Primes à la Euler

Thm: (Euler) Thre are inf. many primes Proof. Suppose P11-., px are all the primes. $\frac{1}{1-\frac{1}{p_i}} = \sum_{\substack{i=0\\ n=0}}^{\infty} \frac{1}{p_i}$ Consider the product $\frac{k}{\prod_{i=1}^{k} \frac{1}{1-i}} = \left(\sum_{\substack{n=0}^{i} \frac{1}{p_i}} \right) \cdots \left(\sum_{\substack{n=0}^{i} \frac{1}{p_k}} \right)$ $\left(1+\frac{1}{p_{1}}+\frac{1}{p_{1}^{2}},\cdots\right)\left(1+\frac{1}{p_{2}}+\cdots\right)--\left(1+\frac{1}{p_{k}}+\cdots\right)$ By FTA, for any inleger, we can

write m= pt---pkek By considering the appropriate terms in Rtls, 10 \Rightarrow RHS $= \sum_{n=1}^{7} \frac{1}{n}$. Contradiction, ble LHS is finite chile ZII diverges! N=1 Euler's groof achielly gives us more info about primes that Euclid's doesn't:

Thon; (Euler) Z I P 7 prime dwer gez



 $\frac{1}{p \leq N} = \frac{1}{p} \left(1 + \frac{1}{p} + \frac{1}{p^2} + \cdots\right)$ $p \leq N = p \geq N$

by any futorization, we can get 1/an ar a kronis The RHS for any MZW

by picking correct exports, tohere m=pi'---picer. Therefore, $\sum_{n \leq N} \frac{1}{n} \leq \prod_{i=1}^{n} \frac{1}{p}$ Now take log: $log(2n) \leq \sum_{n \leq W} log(n)$ $n \leq W$ $p \leq W$ ||21-log(1-17) PEN

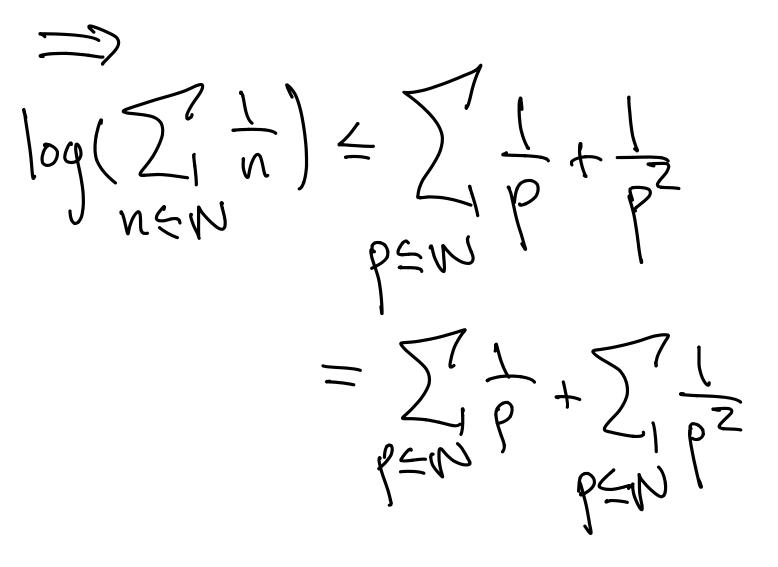
From Calculus, $-\log(t-x)$ $\therefore x + \frac{1}{2}x^{2} + \cdots$ valid for |x| < 1.

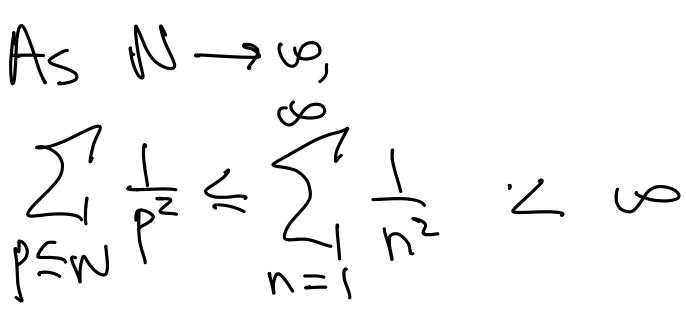
It's not good enough to just use on "approximation", but it turns out for

 $0 \le X \le \frac{1}{2}$ that -1-11 - X + X²

$$=\log(1-x) \leq X + X,$$

So $-\log(1-\frac{1}{p}) \leq \frac{1}{p} + \frac{1}{p^2}$





ZIJ diverges,

Jake away: able to get information about primes by using Calculus. Rimk: Easy, from integral fest Easy, from integral fest I log(x) I len ex $\sum_{i=p}^{T} \frac{1}{p} \propto \log(\log(x))$ $1 \leq p \leq x$ Merken's Second Thin, Harder!

By there in number theory: Shidy behavior of l'anhig finchoirs using long terms/average behavior that calculus is Suited to deal with. E.g. Study primes w/ TT(X). Thm: (Prime Number Thun)

 $\lim_{x \to \infty} \frac{\pi(x)}{\chi} \frac{\chi}{\ln(x)} = 1.$

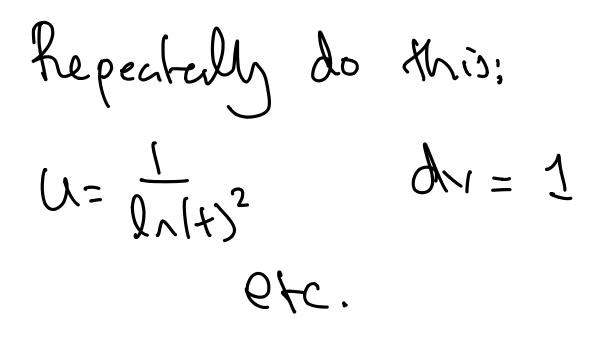
 $x \rightarrow \infty$ i.e. $\pi(x) \sim \chi \ln(x)$ PWT Soups is is roughly # primes <= X $\times/l_{n}(x)$, so T(x) ~ la(x) lee. x ~ la(x) lee. prives have a "density" of L How else to approximate Inlx)-totals of density? Integrate! Gauss Cane up uf a better approximation:

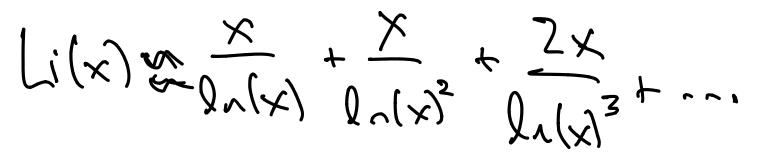
 $Li(x) = \int_{-\infty}^{\infty} \frac{1}{2\pi(4)} dt$ Huernheally, Li(x) Should approximate II(x). In fact, Lilx is a much better approximation to TT(x) than $\frac{x}{ln(x)}$ is! How good? This is the Reman Hypothesis:

There is some Constant C such that $\left| Li(x) - \pi(x) \right| \leq C \int x \log(x)$ tos all × læge enough. i.e. The error between Li(x) and $\Pi(x)$ behaves like Txlog(x). we expect A rough idea: $\begin{pmatrix} C < \frac{131}{984\pi} \end{pmatrix}$ valid for X717657

How to actually estimate Li(x)? Use integration by parts! $Li(x) = \int_{Z} \int_{A(t)} dt$ $u = \frac{1}{2n(t)}$ dv = 1v = t $du = -\frac{1}{t \ln t}$

 $Li(x) = \frac{t}{2n(t)} \Big|_{2}^{x} + \Big(\frac{1}{2n(t)^{2}} dt \Big)$





Some data:

Li(X) $\Pi(x)$ Х 104 1229 5 1229 105 9592 ~ 9571 106 78498 278380 1010 455052511 × 454793911