Problems

Ex: Prove that the product of k consecutive integers is divisible by $k!$

Proof:
Let the integers be $n, n-1, n-2, \ldots, n-(k-1)$. Their product is

$$n(n-1)(n-2)\ldots(n-(k-1))$$

Note that this expression equals

$$\frac{n!}{(n-k)!} \qquad \frac{n!}{k!(n-k)!} = \binom{n}{k}$$

$$= k!\binom{n}{k}$$, so we're done $\square$

Ex: n^3-n is divisible by 6

for all n, b/c $n^3-n = (n-1)n(n+1)$

is the product of 3 consecutive

integers.

Ex: Prove that $((n+1)!+1,\ n!+1)=1$.

Proof:

Let $d = \gcd((n+1)!+1,\ n!+1)$.

Since d is a common divisor,

$d \mid (n+1)!+1$ and $d \mid n!+1$.

So $d \mid ((n+1)!+1) - (n!+1)$

$$(n+1)! - n! = n \cdot n!$$

So $d \mid n \cdot n!$, and $d \mid n! + 1$,

$\Rightarrow \quad d \mid n \cdot n! + n$ and $d \mid n \cdot n!$

So $d \mid n$. From $d \mid n$ and

$d \mid n! + 1$, we see that

$d \mid n \cdot (n-1)! = n!$ and $d \mid n! + 1$

$\Rightarrow \quad d \mid 1 \quad \Rightarrow \quad d = 1 \quad \blacksquare$

Ex: Let a, b be integers with $(a,b) = 1$. Prove that $(a^2, b^2) = 1$.

Proof:

By Bezout's lemma, there are integers x and y such that $ax + by = 1$. We want to show: $a^2u + b^2v = 1$ for some integers u, v.

First try: Square both sides.

$$(ax+by)^2 = 1$$

$$a^2x^2 + \underline{2abxy} + b^2y^2 = 1$$

this doesn't help b/c of the middle term.

Second try: cube both sides.

$$(ax+by)^3 = 1$$

$$\shortparallel$$

$$a^3x^3 + 3a^2x^2by + 3axb^2y^2 + b^3y^3 = 1.$$

$$\shortparallel$$

$$a^2(ax^3 + 3x^2by) + b^2(3axy^2 + by^3) = 1$$

these are integers

So we've found u, v w/

$a^2u + b^2v = 1$, $\Rightarrow$ $(a^2, b^2) \mid 1$

$\Rightarrow$ $(a^2, b^2) = 1$ $\blacksquare$

<u>Ex:</u> Let a, b be integers.

Write $a = p_1^{e_1} \cdots p_k^{e_k}$ $\quad 0 \le e_i, f_i$.

$b = p_1^{f_1} \cdots p_k^{f_k}$

Prove that $\gcd(a,b) =$

$$p_1^{\min\{e_1, f_1\}} \cdots p_k^{\min\{e_k, f_k\}}$$

$$\operatorname{lcm}(a,b) = p_1^{\max\{e_1, f_1\}} \cdots p_k^{\max\{e_k, f_k\}}$$

Proof:

Let d be a common divisor of a and b. By unique factorization, write $d = p_1^{t_1} \cdots p_r^{t_r}$ for some $0 \le t_i$.

Since $d \mid a$, we know that $t_i \le e_i$ for all i.

Similarly, $d \mid b$, so $t_i \le f_i$ for all i.

So $t_i \le \min\{e_i, f_i\}$.

Note that for any integer of the form $p_1^{t_1} \cdots p_k^{t_k}$ w/ $0 \leq t_i \leq \min\{e_i, f_i\}$ that this actually is a divisor of a and b b/c can multiply by missing primes to get either a or b as needed.

To get largest such divisor, maximize t_i.

So $\gcd = p_1^{\min\{e_1, f_1\}} \cdots p_k^{\min\{e_k, f_k\}}$

To get the lcm, let m be a multiple of a and b.

So $a|m$ and $b|m$.

$m = p_1^{s_1} \cdots p_k^{s_k}$ by unique factorization

$0 \leq s_i$.

Since $a|m$, $e_i \leq s_i$ for all
$b|m$ $f_i \leq s_i$ i

So $s_i \geq \max\{e_i, f_i\}$.

Again, note that for

any $s_i \geq \max\{e_i, f_i\}$

that $p_1^{s_1} \cdots p_k^{s_k}$ is a

common multiple, so

we get the least common

mult by minimizing

each exponent.

$$\operatorname{lcm}(a,b) = p_1^{\max\{e_1,f_1\}} \cdots p_k^{\max\{e_k,f_k\}}$$

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We can use this result

to give an alternate

proof of the previous problem:

$$a = p_1^{e_1} \cdots p_k^{e_k} \qquad 0 \leq e_i, f_i$$
$$b = p_1^{f_1} \cdots p_k^{f_k}$$

$$\gcd(a,b) = p_1^{\min\{e_1,f_1\}} \cdots p_k^{\min\{e_k,f_k\}}$$

$$a^2 = p_1^{2e_1} \cdots p_k^{2e_k}$$
$$b^2 = p_1^{2f_1} \cdots p_k^{2f_k}$$

$$\gcd(a^2,b^2) = p_1^{\min\{2e_1,2f_1\}} \cdots p_k^{\min\{2e_k,2f_k\}}$$

and note that $\min\{2e_i, 2f_i\}$

$= 2 \cdot \min\{e_i, f_i\}$ $\blacksquare$

<u>Ex:</u> Find an integer n such that:

- $\frac{n}{2}$ is square
- $\frac{n}{3}$ is a cube
- $\frac{n}{5}$ is a 5^{th} power.

<u>Sol^n:</u>

$$\frac{n}{2} = k^2, \quad \frac{n}{3} = m^3, \quad \frac{n}{5} = l^5$$

for some k, m, l.

$$n = 2k^2 \qquad n = 3m^3 \qquad n = 5l^5$$

We know that n has to be divisible by 2, 3, 5. Let's construct such an n using only these primes.

$$n = 2^e 3^f 5^g \quad \text{for some } e, f, g \geq 1$$

Since $n = 2k^2$, e has be odd.

Since $n = 3m^3$, e has to be divisible by 3

Since $n = 5l^5$, e has to be divisible by 5.

So e has to be an odd multiple of 15, so can take

$e = 15$,

Doing this for f and g:

- f has to be even
- f has to be of the form $3k+1$
- f is divisible by 5

$f = 10$

- g has to be even
- g divisible by 3
- g of the form $5k+1$

$g = 6$

We can take

$$n = 2^{15} \cdot 3^{10} \cdot 5^{6}$$