PELL'S EQUATION AND SQUARE TRIANGULAR NUMBERS

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The sequence of *triangular numbers* are defined via $T_n = \frac{n(n+1)}{2}$. They are named as such because they count the total number of dots in an equilateral triangle with n dots per side. The first few terms of the sequence are given by $1, 3, 6, 10, 15, 21, 28, 36, \ldots$. We see that $T_8 = 6^2$ is a square. Are there any other values of n such that T_n is a square? If so, how many? To answer this question, we take the following approach. We are searching for integers n and k such that $\frac{n(n+1)}{2} = k^2$, or equivalently, $n^2 + n = 2k^2$. We can think of this as a Diophantine problem by trying to find integer points (n, k) with $n, k > 0$ on the hyperbola $x^2+x = 2y^2$. Multiplying through by 2 and completing the square, we can rewrite this as $(2x + 1)^2 - 1 = 8y^2$. Setting $X = 2x + 1$ and $Y = 2y$, we are searching for integer points (X, Y) with $X, Y > 0$ on the hyperbola $x^2 - 2y^2 = 1$. One such solution is the pair $(X, Y) = (3, 2)$, which after solving for x and y yields the integer point $(1, 1)$ on our original curve.

The Diophantine equation $x^2 - 2y^2 = 1$ falls under a special class of Diophantine equations known as *Pell equations*, which are equations of the form $x^2 - Dy^2 = 1$ for D a squarefree positive integer. The remainder of this handout will be dedicated to studying the Pell equation $x^2 - Dy^2 = 1$, to answer our original question.

Set $\mathbb{Z}[\sqrt{D}] = \{a + b\sqrt{D} : a, b \in \mathbb{Z}\}$. The secret to solving our Pell equation will be to study the arithmetic of $\mathbb{Z}[\sqrt{D}]$. For any $\alpha = x + y\sqrt{D}, \beta = x' + y'\sqrt{D} \in \mathbb{Z}[\sqrt{D}]$, we compute

$$\begin{aligned} \alpha + \beta &= (x + x') + (y + y')\sqrt{D} \\ \alpha \cdot \beta &= (xx' + Dyy') + (xy' + x'y)\sqrt{D} \end{aligned}$$

the key point being that $\mathbb{Z}[\sqrt{D}]$ is *closed* under addition and multiplication.

Proposition 1. *Suppose that (x, y) and (x', y') are integer solutions to $x^2 - Dy^2 = 1$. Then the coefficients of $(x + y\sqrt{D})(x' + y'\sqrt{D})$ are integer solutions to $x^2 - Dy^2 = 1$.*

Proof. We have $(x + y\sqrt{D})(x' + y'\sqrt{D}) = (xx' + Dyy') + (xy' + x'y)\sqrt{D}$, so we wish to check that $(xx' + Dyy', xy' + x'y)$ is a solution. We compute $(xx' + Dyy')^2 - D(xy' + x'y)^2 = (x^2x'^2 + 2Dxyx'y' + D^2y^2y'^2) - D(x^2y'^2 + 2xyx'y' + x'^2y^2) = (x^2 - Dy^2)x'^2 - D(x^2 - Dy^2)y'^2 = (x^2 - Dy^2)(x'^2 - Dy'^2) = 1 \cdot 1 = 1$. $\square$

Corollary 0.1. *Suppose that (x, y) is an integer solution to $x^2 - Dy^2 = 1$. For any $n \geq 1$, write $(x + y\sqrt{D})^n = x_n + y_n\sqrt{D}$. Then (x_n, y_n) is a solution to $x^2 - Dy^2 = 1$.*

Proof. For $n = 0$ we trivially have $x_0 = 1$ and $y_0 = 0$ and $(1, 0)$ is a solution. Next, assume $n \geq 1$. The result is true for $n = 1$ by assumption, so assume that (x_n, y_n) is a solution to $x^2 - Dy^2 = 1$. We have $x_{n+1} + y_{n+1}\sqrt{D} = (x + y\sqrt{D})^{n+1} = (x + y\sqrt{D})^n \cdot (x + y\sqrt{D}) = (x_n + y_n\sqrt{D})(x + y\sqrt{D}) = (xx_n + Dyy_n) + (x_ny + xy_n)\sqrt{D}$. This says $x_{n+1} - xx_n - Dyy_n = (x_ny + xy_n - y_{n+1})\sqrt{D}$. Since the left hand side is an integer and the right hand side is an integer times an irrational number, this forces $x_{n+1} = xx_n + Dyy_n$ and $y_{n+1} = x_n y + xy_n$. By the above proposition, (x_{n+1}, y_{n+1}) is a solution. Therefore by induction, (x_n, y_n) is a solution for $n \geq 1$.

□

On our curve are the *trivial solutions* $(\pm 1, 0)$. As long as $(x, y) \neq (\pm 1, 0)$ then the various powers $(x + y\sqrt{D})^n$ are all *distinct*, and give rise to infinitely many integer solutions to $x^2 - Dy^2 = 1$. It turns out we can say something about the *structure* of the solutions. We will call a solution *positive* if both coordinates are positive.

Proposition 2. *Let (x, y) be a positive solution to $x^2 - Dy^2 = 1$, and let (x', y') be any other solution. Then $x + y\sqrt{D} < x' + y'\sqrt{D}$ if and only if $x < x'$ and $y < y'$.*

Proof. The backwards direction is obvious, so we need only to prove the forward direction. Suppose that (x, y) is a positive solution and (x', y') is any solution with $x+y\sqrt{D} < x'+y'\sqrt{D}$. Since (x, y) is positive, we have $x, y \geq 1$ so $x' + y'\sqrt{D} > x + y\sqrt{D} > 1$. Inverting says $0 < x' - y'\sqrt{D} < 1$, so $x' + y'\sqrt{D} > x' - y'\sqrt{D}$. This says $2y'\sqrt{D} > 0$, so $y' > 0$. Since $x' - y'\sqrt{D} > 0$, this says $x' > y'\sqrt{D} \geq \sqrt{D} > 1$, so (x', y') is a positive solution. Now from $x + y\sqrt{D} < x' + y'\sqrt{D}$, inverting says $x - y\sqrt{D} > x' - y'\sqrt{D}$, so $(x + x') + (y - y')\sqrt{D} < (x + x') + (y' - y)\sqrt{D}$, which yields $y' > y$. This then give $x^2 = 1 + Dy^2 < 1 + Dy'^2 = x'^2$, and since $x, x' > 0$ taking a square root tells us $x < x'$. □

If we find a positive solution (x, y) with y minimal, then in fact, x is minimal as well. This is because if (x', y') is any other solution, we have $x^2 = 1 + Dy^2 < 1 + Dy'^2 = x'^2$, and so once more, because (x, y) is positive this means (x', y') is positive, so $x < x'$. We will call a solution to $x^2 - Dy^2 = 1$ with x, y minimal the *fundamental solution* to $x^2 - Dy^2 = 1$.

Theorem 0.2. *Let (x_1, y_1) be the fundamental solution to $x^2 - Dy^2 = 1$. If (x, y) is any other positive solution to $x^2 - Dy^2 = 1$, then $x = x_n$ and $y = y_n$ for some $n \geq 1$, where $(x_1 + y_1\sqrt{D})^n = x_n + y_n\sqrt{D}$.*

Proof. Since (x_1, y_1) is a positive solution, we have $x_1 + y_1\sqrt{D} > 1$, so $(x_1 + y_1\sqrt{D})^n \to \infty$ as $n \to \infty$. Therefore we can find N such that $(x_1 + y_1\sqrt{D})^{N+1} > x + y\sqrt{D} \geq (x_1 + y_1\sqrt{D})^N$. Dividing through says $1 \leq (x + y\sqrt{D})(x_1 + y_1\sqrt{D})^{-N} < x_1 + y_1\sqrt{D}$. Write $(x + y\sqrt{D})(x_1 + y_1\sqrt{D})^{-N} = a + b\sqrt{D}$ for some a, b. Proposition 1 and corollary 0.1 says that (a, b) is a solution to Pell's equation, and since $1 \leq a + b\sqrt{D}$, we have that (a, b) is a *positive* solution by the proof of proposition 2. Therefore, from $a + b\sqrt{D} < x_1 + y_1\sqrt{D}$, applying proposition 2 says $a < x_1$ and $b < y_1$, which contradicts that (x_1, y_1) is the fundamental solution. Therefore, $x + y\sqrt{D} = (x_1 + y_1\sqrt{D})^N$ for some N. □

We have not yet shown that a Pell equation even has a non-trivial solution. As one expects, it does, but this is irrelevant for our purposes: we are concerned with a specific Pell equation, and so we will just find the fundamental solution by hand.

Corollary 0.3. *The positive solutions to $x^2 - 2y^2 = 1$ are $\left(\frac{(3+2\sqrt{2})^n + (3-2\sqrt{2})^n}{2}, \frac{(3+2\sqrt{2})^n - (3-2\sqrt{2})^n}{2\sqrt{2}}\right)$ for $n \geq 1$.*

Proof. To find the fundamental solution to $x^2 - 2y^2 = 1$, we look for the solution with minimal y-coordinate. It's immediately obvious that $(3, 2)$ is the fundamental solution, so any positive solution is of the form (x_n, y_n) where $x_n + y_n\sqrt{2} = (3 + 2\sqrt{2})^n$. Inverting says

$x_n - y_n\sqrt{2} = (3-2\sqrt{2})^n$. Set $\alpha = 3+2\sqrt{2}$, and $\beta = 3-2\sqrt{2}$. This says $2x_n = \alpha^n + \beta^n$ and $2\sqrt{2}y_n = \alpha^n - \beta^n$, so solving for x_n and y_n gives the desired form. □

Theorem 0.4. *The positive solutions to $x^2 + x = 2y^2$ are given by*

$$(x,y) = \left(\frac{(3+2\sqrt{2})^n + (3-2\sqrt{2})^n - 2}{4}, \frac{(3+2\sqrt{2})^n - (3-2\sqrt{2})^n}{4\sqrt{2}}\right)$$

Proof. The above corollary tells us the positive solutions (X,Y) to $x^2 - 2y^2 = 1$. As determined in the introduction, we have $x = \frac{X-1}{2}$ and $y = \frac{Y}{2}$. □

We've determined that there are infinitely many such triangular numbers that are squares. Plugging the first few values of n into the coordinates above, we see the first few square triangular numbers are $T_1 = 1^2, T_8 = 6^2, T_{49} = 35^2, T_{288} = 204^2$, and $T_{1681} = 1189^2$.