DECIMAL EXPANSIONS

TIM SMITS

In elementary school, you learn that a rational number x between 0 and 1 has either a finite decimal expansion $x = .c_1c_2 \dots c_d$, or is eventually *periodic*, that is $x = .b_1b_2 \dots b_m\overline{c_1c_2 \dots c_d}$. The goal of this handout will be to explain this phenomenon, and determine algorithms for determining decimal expansions. Surprisingly, the key to this will be Euler's theorem:

Theorem 0.1. *Let a, m be positive integers with $(a, m) = 1$. Then $a^{\varphi(m)} \equiv 1 \bmod m$.*

We'll start by showing that rational numbers are precisely those with eventually periodic decimal expansions.

Theorem 0.2. *Let x be a real number with $0 < x < 1$. Then x is rational if and only if the decimal expansion of x is eventually periodic.*

Proof. First, suppose that x is rational. Write $x = \frac{a}{b}$ with $(a, b) = 1$, and suppose the decimal expansion of x is $x = .c_1c_2c_3 \dots$. Then $10^k x = c_1c_2 \dots c_k.c_{k+1} \dots$. By the division algorithm, write $10^k a = bq_k + r_k$ where $0 \le r_k < b$. Thus, $c_1c_2 \dots c_k.c_{k+1} = 10^k x = \frac{10^k a}{b} = q_k + \frac{r_k}{b}$, so that $q_k = c_1 \dots c_k$ and $\frac{r_k}{b} = .c_{k+1}c_{k+2} \dots$. Since there are only finitely many possible values for r_k, there exist some integers m, n with $m < n$ such that $r_m = r_n$, so that $.c_{m+1}c_{m+2} \dots = .c_{n+1}c_{n+2} \dots$. This says the decimal expansion of x is $.c_1c_2 \dots \overline{c_{m+1} \dots c_n}$.

Next, assume that $x = .b_1b_2 \dots b_m\overline{c_1 \dots c_d}$. Then $10^m x = b_1 \dots b_m.\overline{c_1 \dots c_d}$. Therefore if we can show $\overline{c_1 \dots c_d}$ is rational, we're done, as $10^m x$ will then be an integer plus a rational number, and therefore solving for x says x is rational. Set $y = .\overline{c_1 \dots c_d}$. Then $10^d y = c_1 \dots c_d.\overline{c_1 \dots c_d}$, so $(10^d - 1)y = c_1 \dots c_d$ says $y = \frac{c_1 \dots c_d}{10^d - 1}$, so that y is rational as desired. ∎

Saying the decimal expansion is finite means that the repeating part is a block of 0's, so we really have proved what we wanted: (the repeating block is either all 0's or it isn't). Note that the above proof is *constructive*: given a rational number, it gives us an algorithm for computing it's decimal expansion, and given a decimal expansion, it tells us what rational number it comes from.

Example 0.3. Let $x = .11\overline{123}$. Then $100x = 11.\overline{123}$, so we need to compute $y = .\overline{123}$. We have $1000y = 123.\overline{123}$, so $999y = 123$ says $y = \frac{123}{999}$. Therefore, $100x = 11 + \frac{123}{999} = \frac{11112}{999}$, so $x = \frac{11112}{99900} = \frac{926}{8325}$.

Example 0.4. Let $x = \frac{1}{303}$. To compute the decimal expansion of x, we follow the proof:

$$\begin{aligned}
10^1 \cdot 1 &= 303 \cdot 0 + 10 \\
10^2 \cdot 1 &= 303 \cdot 0 + 100 \\
10^3 \cdot 1 &= 303 \cdot 3 + 91 \\
10^4 \cdot 1 &= 303 \cdot 33 + 1 \\
10^5 \cdot 1 &= 303 \cdot 330 + 10
\end{aligned}$$

We have found two integers m and n with $r_m = r_n$, namely, $m = 1$ and $n = 5$. This says $x = .c_1\overline{c_2c_3c_4c_5}$. We can now read off the digits by looking at the remainders:

$$\begin{aligned}
c_1 = q_1 &= 0 \\
10c_1 + c_2 = q_2 &= 0 \implies c_2 = 0 \\
100c_1 + 10c_2 + c_3 = q_3 &= 3 \implies c_3 = 3 \\
1000c_1 + 100c_2 + 10c_3 + c_4 = q_4 &= 33 \implies c_4 = 3 \\
10000c_1 + 1000c_2 + 100c_3 + 10c_4 + c_5 = q_5 &= 330 \implies c_5 = 0
\end{aligned}$$

Therefore, $x = .0\overline{0330} = .\overline{0033}$.

This algorithm is not terribly useful: the output for $\frac{1}{303}$ had an initial non-repeating block, however we could actually write it as a purely repeating decimal!

Our next goal will be to determine whether a rational number $x = \frac{a}{b}$ has a finite decimal expansion, or an eventually periodic decimal expansion (with non-zero repeating block), and to determine a better algorithm for computing the decimal expansion. The key step in our proof was that there are finitely many remainders upon division by b, so that $r_m = r_n$ for some integers $m < n$. Translated into a statement about modular arithmetic, there are integers m, n such that $10^m \equiv 10^n \bmod b$. If 10 is invertible mod b, this is the same thing as saying $10^{n-m} \equiv 1 \bmod b$, so that there is a solution to $10^d \equiv 1 \bmod b$.

For any a, m with $(a, m) = 1$, Euler's theorem tells us that $a^{\varphi(m)} \equiv 1 \bmod m$. This says there is some integer d with $a^d \equiv 1 \bmod m$. We give the smallest such d a special name:

Definition 0.5. Let $(a, m) = 1$. The **order of a mod m**, denoted $\text{ord}_m(a)$ is the smallest integer d such that $a^d \equiv 1 \bmod m$.

Example 0.6. $\text{ord}_{100}(7) = 4$ because $7^4 \equiv 1 \bmod 100$ and 4 is the smallest such integer with this property.

The following fact about orders will be very useful for us. For a proof, see the week 5 discussion notes.

Proposition 1. *$a^k \equiv 1 \bmod m$ if and only if $k \mid \text{ord}_m(a)$.*

We'll first tackle the case where x has a *purely periodic* decimal expansion, i.e. $x = .\overline{c_1 \dots c_d}$.

Theorem 0.7. *Let $x = \frac{a}{b}$ with $(a, b) = 1$ be a rational number between 0 and 1. Then the decimal expansion of x is purely periodic if and only if $(10, b) = 1$. In particular, the period length is given by $\text{ord}_b(10)$.*

Proof. First, suppose that $x = .\overline{c_1 \dots c_d}$ is purely periodic. Then $10^d x = c_1 c_2 \dots c_d.\overline{c_1 \dots c_d}$, so $x = \frac{c_1 \dots c_d}{10^d - 1}$. Since $10^d - 1 \equiv 1 \bmod 10$, the denominator remains co-prime to 10 even after canceling common factors with the numerator.

Now suppose that $(10, b) = 1$, and let $d = \mathrm{ord}_b(10)$. Then $10^d \equiv 1 \bmod b$, so $10^d - 1$ is a multiple of b. Write $10^d - 1 = bk$ for some k, so that $x = \frac{a}{b} = \frac{ak}{bk} = \frac{ak}{10^d - 1}$. Since $\frac{a}{b} < 1$ we have $ak < 10^d - 1$, so the decimal expansion of ak requires at most d digits. Write $ak = c_1 \dots c_d$, so $x = \frac{c_1 \dots c_d}{10^d - 1} = .\overline{c_1 \dots c_d}$.

Finally, suppose that x can be written as a repeating decimal with block length ℓ. The above argument shows that x can be written as a fraction with denominator $10^\ell - 1$, so that $10^\ell \equiv 1 \bmod b$. This means $d = \mathrm{ord}_b(10)$ satisfies $d \mid \ell$, so $d \leq \ell$. Since we've shown that x *can* be written as a repeating decimal with block length d, this shows it is the minimal such length, and therefore the period. □

Now, we'll tackle when x has a finite decimal expansion.

Theorem 0.8. *Let $x = \frac{a}{b}$ with $(a, b) = 1$ be a rational number between 0 and 1. Then the decimal expansion of x is finite if and only if the only possible prime factors of b are 2 and 5.*

Proof. First, suppose that $b = 2^e 5^f$ for some e, f. Let $d = \max\{e, f\}$. Then $10^d x = ka$ for some integer k, so $x = \frac{ka}{10^d}$ says the decimal expansion of x is an integer with some number of zeros before it, i.e. is finite.

Next, suppose that x has a finite decimal expansion, $x = .c_1 c_2 \dots c_d$. Then $10^d x = c_1 \dots c_d$, so $x = \frac{c_1 \dots c_d}{10^d}$. Canceling common factors from the numerator and denominator to reduce to common form, this says the only prime factors of the b must divide 10^d, i.e. must be 2 or 5. □

If we combine the two statements, we get the following theorem:

Theorem 0.9. *Let $x = \frac{a}{b}$ where $(a, b) = 1$ be a rational number between 0 and 1. Depending on the prime factorization of b, exactly one of the following holds:*

(1) *x has a finite decimal expansion, if and only if $b = 2^e 5^f$ for some e, f not both 0.*
(2) *x is purely periodic with period length $\mathrm{ord}_b(10)$ if and only if $(10, b) = 1$.*
(3) *x is eventually periodic with an initial non-repeating block if and only if $(10, b) \neq 1$ and b is divisible by a prime other than 2 or 5. If $b = 2^e 5^f b'$, then the period length is $\mathrm{ord}_{b'}(10)$.*

Proof. Everything is immediate from what we have done so far, except the claim about the period length in statement (3). Let $k = \max\{e, f\}$, then $10^k x = \frac{ma}{b'}$ for some integer m. Writing $ma = b'q + r$, with $0 \leq r < b'$, we have $\frac{ma}{b'} = q + \frac{r}{b'}$. Since $(10, b') = 1$, this says $\frac{r}{b'}$ is periodic of length $\mathrm{ord}_{b'}(10)$, so $10^k x$ has a purely periodic fractional part of period $\mathrm{ord}_{b'}(10)$. Dividing by 10^k shifts the decimal point left by k places, so that x has an initial non-repeating block (the digits of q) followed by a repeating block. □

By using modular arithmetic, we can improve the algorithm from our earlier example.

Example 0.10. Let $x = \frac{1}{303}$. Since $(10, 303) = 1$, the above theorem says x is purely periodic with period length $d = \text{ord}_{303}(10)$. By Euler's theorem, $\varphi(303) = \varphi(3)\varphi(101) = 200$, so $d \mid 200$. One can check manually that $10^4 \equiv 1 \bmod 303$, so $d = 4$. We have $10^4 - 1 = 303 \cdot 33$, so $\frac{1}{303} = \frac{33}{10^4-1} = .\overline{0033}$.

Example 0.11. Let $x = \frac{1}{200}$. Then $200 = 2^3 \cdot 5^2$, so x has a finite decimal expansion. We have $\max\{3, 2\} = 3$, so $1000x = 5$. Shifting the decimal to the left 3 places says $x = .005$.

Example 0.12. Let $x = \frac{926}{8325}$. We have $8325 = 3^2 \cdot 5^2 \cdot 37$, so x has an initial non-repeating block following by a repeating block. To compute the decimal expansion of x, we'll use a combination of the previous two methods. Multiply x by 100, so that $100x = \frac{926 \cdot 4}{9 \cdot 37} = \frac{3704}{333} = 11 + \frac{41}{333}$. Since $(10, 333) = 1$, $\frac{41}{333}$ is purely periodic. We find $\varphi(333) = 216$ so $d = \text{ord}_{333}(10) \mid 216$. One can check that $10^3 \equiv 1 \bmod 333$, so $d = 3$. We have $10^3 - 1 = 333 \cdot 3$, so $\frac{1}{333} = \frac{3}{10^3-1}$. This says $\frac{41}{333} = \frac{123}{10^3-1}$, so $\frac{41}{333} = .\overline{123}$. Therefore, $100x = 11.\overline{123}$, so shifting the decimal two places left gives $x = .11\overline{123}$.

Example 0.13. As a final example, the fractions $\frac{1}{7}, \frac{2}{7}, \ldots, \frac{6}{7}$ are all purely periodic because $(10, 7) = 1$. One can check that $\text{ord}_7(10) = 6$, so that each of these fractions are periodic of length 6. As you're probably aware, these fractions are *cyclic* shifts of each other:

$$\frac{1}{7} = .\overline{142857} \quad \frac{2}{7} = .\overline{285714} \quad \frac{3}{7} = .\overline{428571}$$
$$\frac{4}{7} = .\overline{571428} \quad \frac{5}{7} = .\overline{714285} \quad \frac{6}{7} = .\overline{857142}$$

Why does this happen? Since $\text{ord}_7(10) = 6$, the powers $10^k \bmod 7$ for $1 \le k \le 6$ must all be distinct. Since there are 6 non-zero elements mod 7, we actually hit all of them: for any $1 \le m \le 6$, there is k such that $m \equiv 10^k \bmod 7$. This says that $\frac{m}{7}$ and $\frac{10^k}{7}$ have the same fractional part, but the latter we can compute by just shifting the decimal point! To illustrate this, suppose we wanted to compute $\frac{5}{7}$. One can check that $10^5 \equiv 5 \bmod 7$, so $\frac{5}{7}$ and $\frac{10^5}{7}$ have the same fractional part. From $\frac{1}{7} = .\overline{142857}$, we find $\frac{10^5}{7} = 14285.\overline{714285}$ (we know this repeats because the period length is *independent* of the numerator, so the first 6 digits must be the repeating block), which gives $\frac{5}{7} = .\overline{714285}$.

Integers for which $\frac{1}{n}$ have the cyclic shifting property listed above are quite rare: it turns out, these are precisely the integers n such that $\text{ord}_n(10) = \varphi(n)$. The fractions with $n \le 100$ for which this holds are $\frac{1}{7}, \frac{1}{17}, \frac{1}{19}, \frac{1}{23}, \frac{1}{29}, \frac{1}{47}, \frac{1}{49}, \frac{1}{59}, \frac{1}{61}, \frac{1}{97}$. In general, finding such an n where this condition holds is very hard!