Worksheet 20

Problems marked with a $(*)$ are eligible for presentations. (You must present all parts.)

1. What happens when the derivative condition in Hensel's lemma is not met? Suppose that $f(c) \equiv 0 \bmod p$ and $f'(c) \equiv 0 \bmod p$. Let r be a lift of $c \bmod p$ to a congruence class mod p^2. Prove there are only two possibilities: either all such lifts are roots of $f(x) \bmod p^2$, or no such lift is a root mod p^2.

2. $(*)$ Let a be an integer and p a prime such that $\gcd(a, p) = 1$. Set $f(x) = ax - 1$.

   (a) Use Hensel's lemma to find a recursive formula for the solution to $ax \equiv 1 \bmod p^k$ for $k \geq 1$.

   (b) Use your formula to compute the inverse of $5 \bmod 17^3$.

3. $(*)$ Let $f(x) = x^3 - 1$. Note that the solutions to $f(x) \equiv 0 \bmod 7$ are $x \equiv 1, 2, 4 \bmod 7$.

   (a) Use Hensel's lemma to compute the lift of $2 \bmod 7$ that solves $f(x) \equiv 0 \bmod 7^4$.

   (b) Without doing Hensel's lemma a second time, explain how one could easily find the lift of $4 \bmod 7$ that solves $f(x) \equiv 0 \bmod 7^4$. (*Hint: how does* $x^3 - 1$ *factor?*)

4. $(*)$ Let $f(x) = x^4 + 2x + 36$. Find all solutions to the congruence equation $f(x) \equiv 0 \bmod 4375$ (Note: $4375 = 5^4 \cdot 7$).

5. Find all solutions to $f(x) \equiv 0 \bmod 2^n$ for $n \geq 1$, where $f(x) = x^2 + 2x + 5$.

6. Make a diagram showing which lifts of $f(x) = x^4 - 7x^3 + 2x^2 + 2x + 1$ are solutions to $f(x) \equiv 0 \bmod 3^n$ for $n = 1, 2, 3, 4, 5$. Any patterns?