Worksheet 13

Problems marked with a $(*)$ are "key results".

1. $(*)$(Chinese Remainder Theorem) If $m, n > 1$ are co-prime, prove the map $f : \mathbb{Z}/mn\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ given by $f([a]_{mn}) = ([a]_m, [a]_n)$ is a bijection. Here, $[a]_k$ denotes the congruence class of a mod k. *(You should first make sure this is actually a function, i.e. that it doesn't depend on the choice of representative for the congruence class!)*

Our first application of the Chinese Remainder Theorem will be studying the function $\varphi(n)$.

2. $(*)$ Suppose that $m, n > 1$ are relatively prime integers.
   (a) Prove that the restriction f' of f to the units mod mn gives a bijection $f' : (\mathbb{Z}/mn\mathbb{Z})^\times \to (\mathbb{Z}/m\mathbb{Z})^\times \times (\mathbb{Z}/n\mathbb{Z})^\times$.
   (b) Explain how this shows $\varphi(mn) = \varphi(m) \cdot \varphi(n)$.

3. The above problem shows that $\varphi(mn) = \varphi(m)\varphi(n)$ when m and n are relatively prime. How much do they differ when they aren't? Compute $\varphi(mn)$ and $\varphi(m)\varphi(n)$ for all m, n with $mn < 20$. Any conjectures?

4. (a) Using your formula from Monday's worksheet and problem 2, write down a formula for $\varphi(n)$ in terms of the prime factorization of n.
   (b) Compute $\varphi(100)$, $\varphi(360)$, and $\varphi(7000)$.

5. (a) Show that $\varphi(n)$ is even for $n > 2$.
   (b) For which values of n is $\varphi(n)$ divisible by 4? Divisible by 8? *(The formula from 4(a) will be helpful here.)*

6. For a function $f : \mathbb{N} \to \mathbb{Z}$ we write $\sum_{d|n} f(d)$ to denote the sum over all positive divisors of n.

   For example, if $n = 6$ the sum is $f(1) + f(2) + f(3) + f(6)$. Compute $\sum_{d|n} \varphi(d)$ for n up to 12.

   Any conjectures?