Worksheet 12

Problems marked with a ($*$) are "key results".

1. Which of the following rules define functions? Of the functions, which are injective? Surjective? Bijective? Can you find an inverse function, and if so, what is it?
   (a) $f : \mathbb{Z} \to \mathbb{Z}/15\mathbb{Z}$ where $f(n) = [n]$.
   (b) $f : \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/3\mathbb{Z}$ where $f([n]) = [n]$.
   (c) $f : \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$ where $f([n]) = [n]^{-1}$. *(Remember, p always denotes a prime!)*
   (d) $f : \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ where $f([n]) = [n]$.
   (e) $f : \mathbb{Q} \to \mathbb{Q}$ where $f(\frac{a}{b}) = \frac{a+1}{b+1}$.
   (f) $f : \mathbb{Z}/11\mathbb{Z} \to \mathbb{Z}/11\mathbb{Z}$ where $f([n]) = \sqrt[3]{[n]}$.
   (g) $f : \mathbb{Z} \to (\mathbb{Z}/p\mathbb{Z})[x]$ where $f(n) = [n]x$.
   (h) $f : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ where $f([n]) = [2n]$. *(Here, $n > 1$ is arbitrary. Your answer might depend on what it is!)*

2. The *Cartesian product* of two sets X and Y, denoted $X \times Y$, is defined by $X \times Y = \{(x, y) : x \in X, y \in Y\}$.
   (a) Let $X = \{a, b, c, d, e\}$ and $Y = \{1, 2, 3\}$. Explicitly write down all the elements of $X \times Y$.
   (b) In general, when X and Y are finite, how many elements does $X \times Y$ have?
   (c) Write down all the elements of $\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ for various values of m and n. Is there a way to make sense of addition in $\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$? What about multiplication?

3. ($*$) Let X and Y be finite sets and $f : X \to Y$ a function. What can you say about the sizes of X and Y when f is injective? Surjective? Bijective? Give proofs in each case.

4. (Hard) When X and Y are not finite sets, it becomes much harder to come up with a notion of "size".
   (a) Is there a bijection $f : \mathbb{N} \to \mathbb{Z}$? If so, find one. If not, explain why.
   (b) Is there a bijection $f : \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$? If so, find one. If not, explain why.

5. ($*$) Prove that a function $f : X \to Y$ is invertible if and only if f is a bijection.

6. Suppose that $f : X \to Y$ and $g : Y \to Z$ are functions, and let $g \circ f : X \to Z$ be their composition. If f and g are both injective, must $g \circ f$ be injective? What can you say about $g \circ f$ when f and g are surjective? Bijective?