Worksheet 7

Problems marked with a $(*)$ are "key results".

1. $(*)$ Let $n > 1$ be an integer. Suppose that $n = p_1 p_2 \cdots p_r = q_1 q_2 \cdots q_s$ are two factorizations of n into a product of primes with $p_1 \leq p_2 \leq \ldots \leq p_r$ and $q_1 \leq q_2 \leq \ldots \leq q_s$. Prove that $r = s$ and that $p_i = q_i$ for all $1 \leq i \leq r$.

The remaining problems are all consequences of unique factorization. You should only start them after you've finished up problem 1!

2. $(*)$ Write $a = \displaystyle\prod_{p \text{ prime}} p^{e_p}$ and $b = \displaystyle\prod_{p \text{ prime}} p^{f_p}$ as products over all primes. Prove that $a \mid b$ if and only if $e_p \leq f_p$ for all p.

3. $(*)$ Let $n > 1$ be an integer. Prove there are integer d, b such that $a = db^2$ with d square-free (i.e. d is a product of distinct primes).

4. Let a, b be integers, and let $n \geq 1$. Prove that $a^n \mid b^n$ if and only if $a \mid b$.

5. Show that $\sqrt{12}$ is irrational. What other square roots can you show are irrational? Any conjectures?

6. Show that $\log_3(2)$ is irrational. What other logarithms can you show are irrational? Any conjectures?