Worksheet 6

Problems marked with a $(*)$ are "key results".

1. (a) If you haven't already, prove Euclid's lemma from the previous worksheet.
   (b) $(*)$(Generalized Euclid's lemma) Suppose that p is prime and $a_1, a_2, \ldots, a_n \in \mathbb{Z}$ such that $p \mid a_1 a_2 \ldots a_n$. Prove that $p \mid a_i$ for some i.
   (c) Prove the converse to part (a), that if p is a positive integer with the property that for any $a, b \in \mathbb{Z}$, $p \mid ab \implies p \mid a$ or $p \mid b$, then p is a prime.

   This gives an alternate definition of what it means for an integer p to be prime: p is prime if and only if for all integers a, b, $p \mid ab \implies p \mid a$ or $p \mid b$.

2. $(*)$ Prove that every integer $n > 1$ can be factored as a product of primes.

3. (a) Factor 12! into a product of primes.
   (b) Find the exponent of 2 in the prime factorization of 37!. Can you find the exponent of other primes? How might this lead to a factorization of n!?

4. Write down the number of positive divisors of n for $n = 1, \ldots, 30$. Do you notice any patterns? For which values of n are there an odd number of divisors? Any conjectures?