Worksheet 5

Problems marked with a ($*$) are "key results".

1. (Euclid's lemma) Suppose that p is a prime. Prove that for any integers a, b that $p \mid ab \implies p \mid a$ or $p \mid b$.

2. ($*$) Prove that every integer $n > 1$ has a prime divisor.

3. Let p_n denote the n-th prime number, so $p_1 = 2$, $p_2 = 3$, and so on. Define $A_n = p_1 p_2 \dots p_n + 1$.

   (a) Use WolframAlpha or equivalent to compute and factor A_n for various values of n. Find an n for which A_n is composite.

   (b) Let $S_n = \{p_1, p_2, \dots, p_n\}$ be the set containing the first n primes. How do the prime factors of A_n relate to the elements of S_n? Make a conjecture!

   (c) ($*$) Prove that there are infinitely many primes.

The next few problems are all concerned with various methods of finding primes.

4. Write down the primes less than 100 without using a calculator, and think about how you decide whether or not each number you select is prime or not.

5. (a) ($*$) Prove that if $n > 1$ is composite, then n has a prime factor p with $p \leq \sqrt{n}$.

   (b) Explain how the previous part gives a test for whether or not an integer is prime. Then, use your method to determine if 101 is prime or not.

6. (Sieve of Eratosthenes) Write down all natural numbers from 1 to 100, perhaps on a 10×10 array. Circle the number 2, the smallest prime. Cross off all numbers divisible by 2. Circle 3, the next number that is not crossed out. Cross of all larger numbers that are divisible by 3. Continue to circle the smallest number that is not crossed out and cross out all its multiples. Repeat. Why are the circled numbers all the primes less than 100?