Worksheet 2

Problems marked with a (*) are "key results".

- 1. The statements listed below are either true or false. If they are true, you should prove them. If they are false, give a counterexample to disprove them. Additionally, "salvage" the statement, i.e. modify it to create a new true statement. The universe of your salvaged statement should be as close to the original as possible. Then, prove your salvage!
 - (a) (*) For any $a, b \in \mathbb{Z}$, if $a \mid b$ then $a \leq b$.
 - (b) (*) For any $a, b \in \mathbb{Z}$, if $a \mid b$ and $b \mid a$, then a = b.
 - (c) For any $a, b \in \mathbb{Z}$, if ab = 0 then a = 0 or b = 0.
 - (d) For any $a, b, c \in \mathbb{Z}$, if ac = bc, then a = b.
- 2. (Hard) On the last worksheet, you were tasked with using the axioms of \mathbb{Z} to prove that 1 is a positive integer. Go one step further: prove that 1 is the *smallest* positive integer.
- 3. Apply the division algorithm to each of the following pairs of integers a, b. That is, for each one, find q, r such that a = bq + r with $0 \le r < b$.
 - (a) a = 47, b = 13
 - (b) a = 823, b = 48
 - (c) a = -6257, b = 316
- 4. (*) Prove the uniqueness statement of the division algorithm. That is, prove that if there are integers $q, q' \in \mathbb{Z}$ and $r, r' \in \mathbb{Z}$ with $0 \leq r, r' < b$, satisfying a = bq + r and a = bq' + r', that q = q' and r = r'.
- 5. (*) Let $a, b \in \mathbb{Z}$ with $b \neq 0$. Let q, r be as in the division algorithm, so that a = bq + r. Prove the following:
 - (a) $b \mid a$ if and only if r = 0.
 - (b) If $c \mid a$ and $c \mid b$, then $c \mid r$.
 - (c) If $c \mid b$ and $c \mid r$, then $c \mid a$.
- 6. Let $a, b \in \mathbb{N}$. Define an iterative process through the division algorithm as follows:

$$a = bq_1 + r_1$$

$$b = r_1q_2 + r_2$$

$$r_1 = r_2q_3 + r_3$$

$$r_2 = r_3q_4 + r_4$$

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Investigate this process for the following pairs of integers:

- (a) a = 517, b = 89
- (b) a = 75, b = 45

- (c) a = 379427895, b = 1002001 (you should use a calculator for the steps here.)
- (d) Any other pairs a, b that your heart desires.

Write down your observations about the sequence $\{r_n\}$. Do you see any patterns? Any conjectures?