Worksheet 21

Problems marked with a $(*)$ are eligible for presentations. (You must present all parts.)

1. $(*)$
   (a) Find all integer solutions to $a^2 + b^2 = 2c^2$.
   (b) Prove there are no integer solutions to $a^2 + b^2 = 3c^2$. What goes wrong when you try to use your method from part (a)?
2. $(*)$ Find all integer solutions to $a^2 - b^2 = c^2$.
3. $(*)$ Find all integer solutions to $a^2 - ab + b^2 = c^2$.
4. The curves you are classifying rational points on in the above examples all fall under a special class of curves. What type of curves are these? Formulate a general procedure for finding rational points on such a curve, and explain why your procedure works!
5. An *elliptic curve* is a curve of the form $y^2 = x^3 + ax + b$ for $a, b \in \mathbb{Q}$, with certain conditions on a and b. The problem of trying to understand rational points on elliptic curves is one of the biggest areas of research in number theory.
   (a) Consider the elliptic curve $y^2 = x^3 + 8$. Two rational points on this curve are given by $(1, -3)$ and $(-7/4, 13/8)$. The line passing through these two points intersects the curve in exactly one other point. Compute the coordinates of this point. (You will probably need WolframAlpha.)
   (b) Without fully doing the computation, explain why the coordinates of this point must be rational.
   (c) Let (p, q) be your rational point from part (a). Try the method of part (a) with the points $(1, -3)$ and (p, q) and verify that you get back the rational point $(-7/4, 13/8)$. Do you have any ideas on how you could produce a new rational point? *(Hint: as with all things geometric, draw some pictures!)*