Math 11N
Homework 4
Due Monday, February 7th, 2022

1. Let $a, b > 1$ be integers. Let

$$a = 2^{e_2} \cdot 3^{e_3} \cdot 5^{e_5} \cdot 7^{e_7} \cdots = \prod_{p \text{ prime}} p^{e_p}$$
$$b = 2^{f_2} \cdot 3^{f_3} \cdot 5^{f_5} \cdot 7^{f_7} \cdots = \prod_{p \text{ prime}} p^{f_p}$$

be the prime factorizations of a and b respectively.

   (a) Find an expression for $\gcd(a, b)$ in terms of the prime factorizations of a and b. Prove that your answer is correct.

   (b) Find an expression for $\operatorname{lcm}(a, b)$ in terms of the prime factorizations of a and b. Prove that your answer is correct.

   (c) Use the results of the previous parts to compute $\gcd(2^4\, 3^8\, 7^5\, 19^1, 2^2\, 3^3\, 7^4\, 11^2\, 13^3)$ and $\operatorname{lcm}(2^3\, 3^2\, 5^9\, 13^3, 2^5\, 5^3\, 7^2\, 13^2\, 17^1)$.

2. (a) Let $a, b > 1$ be integers. Prove that

$$\gcd(a, b) \cdot \operatorname{lcm}(a, b) = ab.$$

   (b) The previous part provides an efficient way of computing the lcm of two positive integers using the Euclidean algorithm. Use this to compute the least common multiple of 3134376 and 17599768.

3. (a) Prove that every odd number is of the form $4k + 1$ or $4k + 3$ for some integer k.

   (b) Prove that any integer of the form $4k + 3$ has a prime factor of the form $4k + 3$.

   (c) Modify the argument that there are infinitely many primes to show that there are infinitely many primes of the form $4k + 3$ *(Hint: consider* $4(p_1 \cdots p_n - 1) + 3$*)*.

4. (a) Prove by induction that every integer $m \geq 2$ is of the form $m = 2x + 3y$ for some integers $x, y \geq 0$.

   (b) Suppose $n > 1$ is an integer such that the exponent of each prime in its factorization is at least two, e.g. $n = 2^3 \cdot 3^5 = 1944$. Prove that there are integers a, b such that $n = a^2 b^3$.

5. We define the function $\tau(n)$ to be the number of positive divisors of n. For example, $\tau(4) = 3$ because 4 has positive divisors $1, 2, 4$ and $\tau(12) = 6$ because it has positive divisors $1, 2, 3, 4, 6, 12$. The goal of this problem is to explicitly compute $\tau(n)$ for any integer n.

(a) Let p be a prime. Compute $\tau(p^e)$ for any $e \geq 1$.

(b) Let m, n be integers with $\gcd(m, n) = 1$. Prove that if $d \mid mn$ then $d = d_1 d_2$ for unique d_1, d_2 with $d_1 \mid n$ and $d_2 \mid m$. Using this, explain why $\tau(mn) = \tau(m)\tau(n)$.

(c) Write down a formula for $\tau(n)$ in terms of the prime factorization of n. Use your formula to compute the number of positive divisors of 10!.

This last problem deal with so-called *local techniques* – zooming in on a specific prime in the factorization and using knowledge of its exponent to gain information. Let

$$n = 2^{e_2} \cdot 3^{e_3} \cdot 5^{e_5} \cdot 7^{e_7} \cdots = \prod_{p \text{ prime}} p^{e_p}$$

be the prime factorization of n. For each prime p, define a function v_p by $v_p(n) = e_p$. For example, $v_3(72) = 2$ because $72 = 2^3 \cdot 3^2$. Phrased in this language, we see that $p \mid n$ if and only if $v_p(n) > 0$.

6. (a) Prove that for integers a, b we have $v_p(ab) = v_p(a) + v_p(b)$ and $v_p(a + b) \geq \min\{v_p(a), v_p(b)\}$. If a, b are relatively prime, what can you say about $v_p(a)$ and $v_p(b)$?

Using the above properties of v_p, prove the following:

(b) Let a, b be relatively prime integers and suppose that $ab = c^k$ for some integers c, k. Prove that a and b are both k-th powers.

(c) Let $a, b, k > 1$ be integers with $\gcd(a, b) = 1$. Prove that $\sqrt[k]{a/b}$ is rational if and only if a and b are both k-th powers .