Math 11N

Homework 2

Due Thursday, January 20th, 2022

1. Suppose that $n > 1$ is an odd integer. Prove that $8 \mid n^2 - 1$.

2. (a) Let x be an integer. Prove by induction that for any integer $n \geq 1$, $(x-1) \mid (x^n - 1)$.

   (b) Let x be an integer. Prove by induction that if $n \geq 1$ is odd, then $(x+1) \mid (x^n+1)$.

3. (a) Prove that for any integer $n \geq 1$, if $2^n - 1$ is prime, then n must be prime.

   (b) Prove that for any integer $n \geq 1$, if $2^n + 1$ is prime, then $n = 2^\ell$ for some integer $\ell \geq 0$. (You may assume for this problem that every integer may be factored into primes).

   (c) Show that the converse of each statement is false (you may use WolframAlpha for computations).

4. In the original statement of the division algorithm, we assumed that a and b are integers with $b > 0$. You will now extend this result to an *arbitrary* non-zero integer b.

   (a) For $a, b \in \mathbb{Z}$ with $b \neq 0$, prove that there exist integers q, r with $a = bq + r$ and $0 \leq r < |b|$. *(Hint: try applying the version you have already proved!)*

   (b) Prove the uniqueness of q and r in this more general version of the theorem. Does the original proof still work?

5. Apply the theorem just proved in the previous problem to each of the following pairs of numbers a, b. That is, for each one, find integers q and r such that

$$a = b \cdot q + r \qquad \text{with } 0 \leq r < |b|.$$

   (a) $a = 47$, $b = -13$

   (b) $a = 956$, $b = -27$

   (c) $a = 29657452$, $b = -4382$

6. Find all rectangles with integer sides with equal area and perimeter.