Math 11N
Homework 1
Due Thursday, January 13, 2022

1. A propositional expression that is always true, for all truth values of the propositional variables contained in it, is called a *tautology*. For each of the following, use a truth table to show that the expression is a tautology.

   (a) $(P \wedge Q) \implies P$

   (b) $((P \implies Q) \wedge P) \implies Q$

   (c) $((P \vee R) \wedge (Q \vee R)) \iff ((P \wedge Q) \vee R)$

2. *How to prove an "either-or" statement.*

   (a) Prove (using a truth table) that $P \vee Q$ is logically equivalent to $(\neg P) \implies Q$, and is also logically equivalent to $(\neg Q) \implies P$.

   (b) Suppose you are studying a type of natural number called *magic numbers*. Use part (a) to explain how you might prove the statement "Every magic number is either even or prime."

3. Each of the following pairs of statements differ only in the order of two quantifiers. For each pair of statements, write down in plain English what the statement is saying, and then determine if each statement is true or false. (No justification needed.)

   (a) $\forall a \in \mathbb{Z} \quad \exists b \in \mathbb{Z} \quad a + b = 0$
   $\exists b \in \mathbb{Z} \quad \forall a \in \mathbb{Z} \quad a + b = 0$

   (b) $\forall u \in \mathbb{R} \quad \exists v \in \mathbb{R} \quad uv = v$
   $\exists v \in \mathbb{R} \quad \forall u \in \mathbb{R} \quad uv = v$

   (c) $\exists x \in \mathbb{R} \quad \forall y \in \mathbb{R} \quad x \leq y$
   $\forall y \in \mathbb{R} \quad \exists x \in \mathbb{R} \quad x \leq y$

   (d) Let H be the set of all humans.
   $\exists c \in H \quad \forall p \in H \quad c$ is a child of p
   $\forall p \in H \quad \exists c \in H \quad c$ is a child of p

The next four problems all state mathematical definitions. For each of these definitions, do the following:

(i) Write the definition in symbolic form.

(ii) Give the negation of the definition, in symbolic form. In your result, any $\neg$ should come *after* the last quantifier, and anything of the form $\neg(P \implies Q)$ should be simplified so it doesn't include a $\implies$.

(iii) Translate the negation that you just wrote in symbols back into English. **This final result should include only the symbols that are in the original definition. No more logical symbols or quantifiers are allowed here.**

Note: By "the definition", I mean only the part coming after the term that's being defined (which I've placed in italics). Everything before that is just context. To help, in these problems, I've placed the part you should negate on a line by itself.

4. An integer n is *odd* if

   there exists an integer k such that $n = 2k + 1$.

5. Let X and Y be sets. A function $f\colon X \to Y$ is called *onto* provided that

   for each $y \in Y$, there is some $x \in X$ for which $f(x) = y$.

6. Let X and Y be sets. A function $f\colon X \to Y$ is called *one-to-one* provided that

   for all x and y in X, if $f(x) = f(y)$ then $x = y$.

7. A positive integer p is called *prime* if

   $p \neq 1$ and the only positive divisors of p are 1 and p.

   (Hint: There's a hidden variable, and thus a hidden quantifier, in this one.)

8. The *binomial coefficient* $\binom{n}{k}$ is defined by

$$\binom{n}{k} = \begin{cases} \dfrac{n!}{k!(n-k)!} & \text{for } 0 \leq k \leq n \\ 0 & \text{otherwise} \end{cases}$$

   (a) Prove that

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

   for all non-negative integers n and k with $0 \leq k \leq n$.
   (*Hint: this proof does not require any induction*)

   (b) Prove by induction that for any integer $n \geq 0$,

$$\sum_{j=0}^{n} \binom{n}{j} = 2^n.$$