Solutions to Homework 1

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1. A propositional expression that is always true, for all truth values of the propositional variables contained in it, is called a *tautology*. For each of the following, use a truth table to show that the expression is a tautology.

(a) $(P \wedge Q) \implies P$

(b) $((P \implies Q) \wedge P) \implies Q$

(c) $((P \vee R) \wedge (Q \vee R)) \iff ((P \wedge Q) \vee R)$

Solution: Omitted, everyone knew how to fill in the truth tables!

2. *How to prove an "either-or" statement.*

(a) Prove (using a truth table) that $P \vee Q$ is logically equivalent to $(\neg P) \implies Q$, and is also logically equivalent to $(\neg Q) \implies P$.

(b) Suppose you are studying a type of natural number called *magic numbers*. Use part (a) to explain how you might prove the statement "Every magic number is either even or prime."

Solution:

(a) Omitted again, everyone knew how to fill in the truth table.

(b) If you want to prove the statement "Every magic number is either even or prime", think of it as the statement "$\forall n \in S, P(n) \vee Q(n)$" where S is the set of magic numbers and the propositional functions $P(n)$ and $Q(n)$ are defined by $P(n) =$ "n is even" and $Q(n) =$ "n is prime". By part (a), you could then instead prove either one of the statements "For all magic numbers, if a magic number is not even, then it is prime" or "For all magic numbers, if a magic number is not prime, then it is even".

3. Each of the following pairs of statements differ only in the order of two quantifiers. For each pair of statements, write down in plain English what the statement is saying, and then determine if each statement is true or false. (No justification needed.)

(a) $\forall a \in \mathbb{Z} \quad \exists b \in \mathbb{Z} \quad a + b = 0$
$\exists b \in \mathbb{Z} \quad \forall a \in \mathbb{Z} \quad a + b = 0$

(b) $\forall u \in \mathbb{R} \quad \exists v \in \mathbb{R} \quad uv = v$
$\exists v \in \mathbb{R} \quad \forall u \in \mathbb{R} \quad uv = v$

(c) $\exists x \in \mathbb{R} \quad \forall y \in \mathbb{R} \quad x \leq y$
$\forall y \in \mathbb{R} \quad \exists x \in \mathbb{R} \quad x \leq y$

(d) Let H be the set of all humans.
$\exists c \in H \quad \forall p \in H \quad c$ is a child of p
$\forall p \in H \quad \exists c \in H \quad c$ is a child of p

Solution:

(a) The first statement claims that for any integer a, you can find an integer b such that $a + b = 0$. This is true, in particular, $b = -a$.
The second statement claims there is a universal choice of integer b such that for any choice of integer a, $a + b = 0$. This is false.

(b) The first statement claims that for any real number u, there is a real number v such that $uv = v$. This is true, in particular, we may take $v = 0$.
The second statement says that there is a universal choice of real number v such that no matter what real number u we pick, we can make $uv = v$. Again, this is true, taking $v = 0$.

(c) The first statement claims that there is a single real number x such that for any real number y, $x \leq y$. That is to say, that there is a real number smaller than any other real number. This is false.
The second statement claims that for any choice of real number y, we can find a real number x such that $x \leq y$, that is to say, that for any choice of real number we can find another one no larger than it. This is of course, true.

(d) The first statement says that there is some person who is a child of everyone. This is false.
The second statements says that every person has a child, which is again of course false.

The next four problems all state mathematical definitions. For each of these definitions, do the following:

(i) Write the definition in symbolic form.
(ii) Give the negation of the definition, in symbolic form. In your result, any $\neg$ should come *after* the last quantifier, and anything of the form $\neg(P \implies Q)$ should be simplified so it doesn't include a $\implies$.
(iii) Translate the negation that you just wrote in symbols back into English. **This final result should include only the symbols that are in the original definition. No more logical symbols or quantifiers are allowed here.**

Note: By "the definition", I mean only the part coming after the term that's being defined (which I've placed in italics). Everything before that is just context. To help, in these problems, I've placed the part you should negate on a line by itself.

4. An integer n is *odd* if

there exists an integer k such that $n = 2k + 1$.

Solution: The symbolic form of the definition is "$\exists k \in \mathbb{Z}, n = 2k + 1$". The negation is "$\forall k \in \mathbb{Z}, n \neq 2k + 1$", which translated back into English says an integer n is *not* odd if for all integers k, $n \neq 2k + 1$.

5. Let X and Y be sets. A function $f : X \to Y$ is called *onto* provided that

for each $y \in Y$, there is some $x \in X$ for which $f(x) = y$.

Solution: The symbolic definition is "$\forall y \in Y, \exists x \in X, f(x) = y$". The negation is "$\exists y \in Y, \forall x \in X, f(x) \neq y$", and the translation back into English says a function is *not* onto if there is a choice of $y \in Y$ such that for all $x \in X$, $f(x) \neq y$.

6. Let X and Y be sets. A function $f : X \to Y$ is called *one-to-one* provided that

for all x and y in X, if $f(x) = f(y)$ then $x = y$.

Solution: The symbolic definition is "$\forall x, y \in X, f(x) = f(y) \implies x = y$". The negation is "$\exists x, y \in X, f(x) = f(y) \wedge x \neq y$", and the translation back into English says a function is *not* one-to-one if there are $x, y \in X$ such that $f(x) = f(y)$ but $x \neq y$.

7. A positive integer p is called *prime* if

$p \neq 1$ and the only positive divisors of p are 1 and p.

Solution: The definition of a prime depends on its divisors: therefore, we have to quantify over all divisors of the integer p. One such way of writing down a definition is as follows: "$(p \neq 1) \wedge (\forall d \in \mathbb{N}, d \mid p \implies d = 1 \vee d = p)$". The negation is given by "$(p = 1) \vee (\exists d \in \mathbb{N}, d \mid p \wedge d \neq 1 \wedge d \neq p)$", and the translation back into English says a positive integer p is *not* prime if it's equal to 1 or if there is a divisor that is not 1 and not p.

8. The *binomial coefficient* $\binom{n}{k}$ is defined by

$$\binom{n}{k} = \begin{cases} \dfrac{n!}{k!(n-k)!} & \text{for } 0 \leq k \leq n \\ 0 & \text{otherwise} \end{cases}$$

(a) Prove that

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

for all non-negative integers n and k with $0 \leq k \leq n$.
(*Hint: this proof does not require any induction*)

(b) Prove by induction that for any integer $n \geq 0$,

$$\sum_{j=0}^{n} \binom{n}{j} = 2^n.$$

Solution:

(a) If $k = 0$ then $\binom{n}{-1} = 0$ by definition, and both $\binom{n+1}{0} = \binom{n}{0} = 1$. Now for $1 \le k \le n$, we have $\binom{n+1}{k} = \frac{(n+1)!}{k!(n+1-k)!}$ and the other two binomial coefficients are $\binom{n}{k-1} = \frac{n!}{(k-1)!(n-k+1)!}$ and $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. Multiply the former by $\frac{k}{k}$ and the latter by $\frac{n-k+1}{n-k+1}$ and add to get $\binom{n}{k-1} + \binom{n}{k} = \frac{n!k}{k!(n-k+1)!} + \frac{n!(n-k+1)}{k!(n-k+1)!} = \frac{n!(n+1)}{k!(n-k+1)!} = \frac{(n+1)!}{k!(n-k+1)!} = \binom{n+1}{k}$ as desired.

(b) For our base case, we check that the desired equality holds for $n = 0$. Pugging in, the left hand side is $\sum_{j=0}^{0} \binom{0}{j} = \binom{0}{0} = 1$ and the right hand side is $2^0 = 1$, so we're good.

Now, suppose that the result holds for some integer k, that is to say, that $\sum_{j=0}^{k} \binom{k}{j} = 2^k$. We wish to prove that it also holds for $k+1$ as well. We have $\sum_{j=0}^{k+1} \binom{k+1}{j} = \sum_{j=0}^{k} \binom{k+1}{j} + \binom{k+1}{k+1} = \sum_{j=0}^{k} \binom{k+1}{j} + 1$, so it remains to evaluate the remaining sum. By part (a), we have $\binom{k+1}{j} = \binom{k}{j-1} + \binom{k}{j}$, so $\sum_{j=0}^{k} \binom{k+1}{j} = \sum_{j=0}^{k} \binom{k}{j-1} + \sum_{j=0}^{k} \binom{k}{j}$. The latter sum is 2^k by induction hypothesis, and the first sum is $\binom{k}{-1} + \sum_{j=1}^{k} \binom{k}{j-1} = \sum_{j=1}^{k} \binom{k}{j-1}$. Shifting the index, we get $\sum_{j=1}^{k} \binom{k}{j-1} = \sum_{j=0}^{k-1} \binom{k}{j}$, which is is $\sum_{j=0}^{k} \binom{k}{j} - \binom{k}{k} = 2^k - 1$ once again induction hypothesis and the definition of binomial coefficients. Putting everything together, we have $\sum_{j=0}^{k+1} \binom{k+1}{j} = (2^k - 1) + 2^k + 1 = 2 \cdot 2^k = 2^{k+1}$ as desired. Therefore by induction, we have proven for all $n \ge 0$ that $\sum_{j=0}^{n} \binom{n}{k} = 2^n$.