**Math 11N**
**Gateway to Mathematics: Number Theory**

**Midterm**

**Directions:** Do the problems below. You have 24 hours to complete this exam, from 10:00 AM PST on Thursday, February 3 to 10:00 AM PST on Friday, February 4, by which time you must scan and upload your exam on Gradescope. You may use any course resources from our Canvas page, but you may not use other internet resources, nor may you discuss the exam with anyone other than me. Do not use methods that have not been covered in class. You may use a calculator. Show your work. Write full sentences when necessary.

Name: ____________________________

UID: ____________________________

| Question | Points | Score |
|---|---|---|
| 1 | 10 | |
| 2 | 10 | |
| 3 | 10 | |
| 4 | 10 | |
| 5 | 10 | |
| Total: | 50 | |

1. (10 pts.) Let $x, y, z$ be consecutive positive odd integers. Prove by induction that $9 \mid x^3 + y^3 + z^3$.

2. (10 pts.) Let $a, b, c \in \mathbb{Z}$ be non-zero. Prove that if $\gcd(a, b) = 1$ and $c \mid (a + b)$, then there exists $x, y \in \mathbb{Z}$ such that $ax + (bc)y = 1$.

3. (a) (5 pts.) Run the Euclidean Algorithm and perform back substitution to find **all** integer solutions to the equation $2310x + 1638y = 3948$.

(b) (5 pts.) Find all *positive* integer solutions to the equation $2310x + 1638y = 3948$.

4. (10 pts.) Let $a, b, c \in \mathbb{Z}$ with $a^7 \mid b^8$ and $b^5 \mid c^4$. Prove that $a \mid c$. (*Hint: you need more than just the definition of divisibility here.*)

5. (a) (5 pts.) For the definition listed below:
   (i) Write the definition in symbolic form.
   (ii) Write the negation in symbolic form. In your result, any $\neg$ should come *after* the last quantifier, and anything of the form $\neg(P \implies Q)$ should be simplified so that it doesn't include a $\implies$.
   (iii) Translate the negation back into plain English.

   An integer $n > 1$ is called *square-free* if

   the only perfect square that divides $n$ is 1.

   (b) (5 pts.) Below is a list of results that you have proven in the course. Starting from the axioms, draw a diagram of arrows that indicates what results you need and in what order they would need to be proven in order to prove the fundamental theorem of arithmetic.

Axioms of $\mathbb{Z}$

Classification of solutions to $ax + by = c$ Euclid's lemma

Integers have prime divisors Bezout's lemma Division algorithm

Basic properties of divisibility Infinitely many primes Euclidean algorithm

Fundamental theorem of arithmetic