Discussion Review Problems

1. Let $V \subset C^\infty(\mathbb{R})$ be the subspace of smooth functions that are 1-periodic, i.e. that satisfy $f(x+1) = f(x)$ for all $x \in \mathbb{R}$. Equip V with the structure of an inner product space by $\langle f, g \rangle = \int_0^1 f(x)g(x)\,dx$. Let $D : V \to V$ denote the derivative map. Compute the adjoint of D. Is D normal? Self-adjoint?

2. Let V be a finite dimensional real inner product space. A self-adjoint operator $T : V \to V$ is called *positive definite* if $\langle T(v), v \rangle > 0$ for all $v \neq 0$.

   (a) Prove that T is positive definite if and only if all eigenvalues of T are positive.

   (b) Prove that if T is invertible, then T^*T is positive definite.

3. Let $T : V \to V$ be a normal operator on a finite dimensional inner product space.

   (a) Prove that $\mathrm{Im}(T) = \mathrm{Im}(T^*)$.

   (b) Prove that $\ker(T^k) = \ker(T)$ and $\mathrm{Im}(T^k) = \mathrm{Im}(T)$ for all $k \geq 0$.

4. Let $T : V \to V$ be a unitary operator on a finite dimensional complex inner product space. Prove that $T = S^2$ for some unitary operator $S : V \to V$.

5. Let $T : V \to V$ be a projection on a finite dimensional inner product space. Show that if $\|T(x)\| \leq \|x\|$ for all $x \in V$, then T is an orthogonal projection.

6. Let $B : M_2(F) \times M_2(F) \to F$ be defined by $B(A, B) = \mathrm{Tr}(AB)$, where $\mathrm{Tr}(X)$ denotes the trace of the matrix X. Prove that B is a non-degenerate bilinear form.

Solutions

1. The adjoint D^* satisfies the relation $\langle D(f), g\rangle = \langle f, D^*(g)\rangle$ for any $f, g \in V$. The left hand side is $\int_0^1 f'(x)g(x)\,dx$. Integrating by parts, this is $-\int_0^1 f(x)g'(x)\,dx = \langle f, -D(g)\rangle$. This tells us that $D^* = -D$. This shows that D is clearly normal and not self-adjoint.

2. (a) Suppose that T is positive definite, and let v be an eigenvector of T with eigenvalue λ. Then $\langle T(v), v\rangle = \lambda\|v\|^2 > 0$ by assumption, so $\lambda > 0$. Conversely, suppose that all eigenvalues of T are positive. Since T is self-adjoint, we can find an orthonormal eigenbasis $\{v_1, \dots, v_n\}$ of V. Then the above shows that $\langle T(v_i), v_i\rangle > 0$ for all i, and so by the bilinearity of the inner product (and orthogonality of the v_i), we have $\langle T(v), v\rangle > 0$ for all v.

   (b) If T is invertible, then $0 < \|T(x)\|^2 = \langle T(x), T(x)\rangle = \langle x, T^*T(x)\rangle = \langle T^*T(x), x\rangle$ for any $x \in V$.

3. (a) I claim that $\ker(T) = \ker(T^*)$. This follows because $T(x) = 0 \iff \|T(x)\|^2 = 0 \iff \langle T(x), T(x)\rangle = 0 \iff \langle x, T^*T(x)\rangle = 0 \iff x, TT^*(x)\rangle = 0 \iff \langle T^*(x), T^*(x)\rangle = 0 \iff \|T^*(x)\|^2 = 0 \iff T^*(x) = 0$. We then have $\mathrm{Im}(T) = \ker(T^*)^\perp = \ker(T)^\perp = \mathrm{Im}(T^*)$ as desired.

   (b) We have two chains of subspaces $\ker(T) \subset \ker(T^2) \subset \dots$ and $\dots \subset \mathrm{Im}(T^2) \subset \mathrm{Im}(T)$, so it sufficies to prove only the other inclusions. To do so, we just prove that $\ker(T^k) = \ker(T)$ for all k, because then by rank nullity one finds that $\mathrm{rank}(T^k) = \mathrm{rank}(T)$ giving the other equality.

   Suppose that $x \in \ker(T^k)$. Then for any $y \in V$, we have $0 = \langle T^k(x), y\rangle = \langle T^{k-1}(x), T^*(y)\rangle$. This says that $T^{k-1}(x) \in \mathrm{Im}(T^*)^\perp = \mathrm{Im}(T)^\perp = \ker(T)$. This means $T^{k-1}(x) \in \mathrm{Im}(T) \cap \mathrm{Im}(T)^\perp$ so $T^{k-1}(x) = 0$. This says that $\ker(T^k) \subset \ker(T^{k-1})$, so that $\ker(T^k) = \ker(T^{k-1})$. An induction argument then shows that $\ker(T^k) = \ker(T)$ for all k.

4. Since T is unitary, we can find an orthonormal basis $\{v_1, \dots, v_n\}$ of V. Let λ_i be the associated eigenvalue of v_i. We know that $|\lambda_i| = 1$. Define $S : V \to V$ by $S(v_i) = \sqrt{\lambda_i}v_i$ (choose either of the two complex square roots). Then clearly $S^2(v_i) = \lambda_i v_i = T(v_i)$, so $T = S^2$. Since $\|S(v_i)\| = |\sqrt{\lambda_i}|\|v_i\| = \|v_i\| = 1$, we see that S is also unitary as desired.

5. We wish to show that $\mathrm{Im}(T)^\perp = \ker(T)$. Suppose that $v \in \mathrm{Im}(T)^\perp$ but that $T(v) \neq 0$. Write $w = T(v)$, and let $x = sw + v$ for some $s \in \mathbb{R}$ to be determined. Note that $w \in \mathrm{Im}(T)$, by the Pythagorean theorem, we have $\|x\|^2 = s^2\|w\|^2 + \|v\|^2$. Note that $T(w) = T^2(v) = T(v) = w$ because T is a projection, so $T(x) = sT(w) + T(v) = (s+1)w$. This says $\|T(x)\|^2 = (s+1)^2\|w\|^2$. By assumption, we then find $(s+1)^2\|w\|^2 \leq s^2\|w\|^2 + \|v\|^2$, so that $2s\|w\|^2 \leq \|v\|^2 - \|w\|^2$. Since v, w are fixed and $w \neq 0$ by assumption, we may choose such an s to make this inequality false. This gives a contradiction, so T is an orthogonal projection as desired.

6. To check that $B$is non-degenerate, we can check that $[B]_\beta$ is invertible where β is a basis of $M_2(F)$. Take $\beta = \{E_{11}, E_{12}, E_{21}, E_{22}\} = \{v_1, v_2, v_3, v_4\}$ to be the standard basis of $M_2(F)$. Then by definition, we have $([B]_\beta)_{ij} = B(v_i, v_j)$. One can check that $[B]_\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ so that $\det([B]_\beta) = -1$ says $[B]_\beta$ is invertible.