Solutions to Midterm 2

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1. Consider the vector space $V = P(\mathbb{R})$ of polynomials with real coefficients, endowed with the unique inner product structure such that the basis $B = \{x^i : i \geq 0\}$ of V is an orthonormal set.

(a) Fix $n \geq 1$ and define $T(g)(x) = x^n g(x)$ for $g \in V$. Find an adjoint T^* of T.

(b) Find the subspace $W = \{w \in V : TT^*(w) = T^*T(w)\}$.

(c) Is W T-invariant? T^*-invariant?

(d) Let $U : V \to V$ be the operator $U(g)(x) = g(x+1)$. Prove that U has no adjoint with respect to the given inner product.

Solution:

(a) I claim that $T^*(x^i) = \begin{cases} x^{i-n} & i \geq n \\ 0 & i < n \end{cases}$. To do so, we just check that T^* satisfies the definition of the adjoint, of which it's sufficient to just check on basis vectors by bilinearity of the inner product. Fix i and let j be arbitrary. If $i \geq n$, we have $\langle T(x^j), x^i \rangle = \langle x^{j+n}, x^i \rangle = \delta_{j+n,i}$, and $\langle x^j, T^*(x^i) \rangle = \langle x^j, x^{i-n} \rangle = \delta_{j,i-n}$. Note that $j+n = i$ if and only if $j = i-n$, so these agree. If $i < n$, we find that $\delta_{j+n,i} = 0$ and $\langle x^j, T^*(x^i) \rangle = \langle x^j, 0 \rangle = 0$. Therefore, T^* is the adjoint of T as desired.

Note: one can deduce such a formula for T^* in a few ways. One such way is as follows: let $\mathbb{R}_0^\infty$ be the space of sequences of real numbers that are eventually 0. Recall the standard basis of this vector space is given by $\{e_i : i \geq 1\}$ where e_i is the vector with 1 in the i-th component and 0 everywhere else. The standard inner product in $\mathbb{R}^n$ extends to an inner product on $\mathbb{R}_0^\infty$ in the obvious way that makes $\{e_i\}$ orthonormal, and so the map $P(\mathbb{R}) \to \mathbb{R}_0^\infty$ given by $x^i \to e_i$ gives an isomorphism of inner product spaces. Under this isomorphism, T is the right shift operator that sends $(a_1, a_2, \ldots, 0, 0, \ldots)$ to $(0, 0, \ldots, 0, a_1, a_2, \ldots, 0, \ldots)$. The adjoint is then given by the left shift operator, which sends $(a_1, a_2, \ldots, 0, \ldots,)$ to $(a_{n+1}, \ldots, 0, \ldots)$, and so converting back to polynomials give the above formula.

(b) Note that $T^*T = I$, so we want to find the largest subspace $W \subset V$ such that $TT^* = I$. We compute that $TT^*(x^i) = \begin{cases} x^i & i \geq n \\ 0 & i < n \end{cases}$. Taking $W = \text{Span}\{x^i : i \geq n\}$ clearly gives the desired result.

(c) Clearly W is T-invariant, but it's not T^*-invariant. For example, $T^*(x^n) = 1$ is not in W.

(d) Suppose that U had an adjoint. Then for any i, we would have $\langle U(x^i), 1 \rangle = \langle x^i, U^*(1) \rangle$. Since $U(x^i) = (x+1)^i$ has constant term 1, we see $1 = \langle U(x^i), 1 \rangle = \langle x^i, U^*(1) \rangle$. The previous expression is the coefficient of x^i in $U^*(1)$, which gives a contradiction because $U^*(1)$ has non-zero coefficients only up until it's degree.

2. Let V be a finite dimensional complex inner product space. Let $a \in \mathbb{C}$ and let $T : V \to V$ be a non-zero linear operator on V such that $T^* = aT$.

(a) Show that $|a| = 1$.

(b) Prove that every eigenvalue of T is on a line ℓ in $\mathbb{C}$ through the origin, and find the angle $\phi \in [0, \pi)$ between the x-axis and ℓ, going from the x-axis to ℓ in the counterclockwise direction.

(c) Show that T is a product of a self-adjoint operator and a unitary operator.

Solution:

(a) Note that $T = (T^*)^* = (aT)^* = \bar{a}T^* = |a|^2 T$. Since T is non-zero, there is $v \in V$ such that $T(v) \neq 0$, so $T(v) = |a|^2 T(v)$ says $(|a|^2 - 1)T(v) = 0$. This forces $|a| = 1$ as desired.

(b) Write $a = e^{i\theta}$. Then let λ be an eigenvalue of T with eigenvector v. Note that $T^* = aT$ means that T is normal, so T^* has v as an eigenvector as well, with eigenvalue $\bar{\lambda}$. This says $\bar{\lambda} = \lambda e^{i\theta}$. Writing $\lambda = re^{i\phi}$ for some r, ϕ, we have $re^{-i\phi} = re^{i(\theta+\phi)}$. This tells us that $-\phi = \theta + \phi$ (up to a multiple of 2π), so that $\phi = -\theta/2$. Since $\theta \in [0, 2\pi)$ converting ϕ to an angle in $[0, \pi)$ gives $\phi = \pi - \theta/2$. This says every eigenvalue of T lies on the line $\phi = \pi - \theta/2$.

(c) Since T is normal, it's orthogonally diagonalizable. Let $\{v_1, \dots, v_n\}$ be an orthonormal eigenbasis of V, so that $T(v_i) = \lambda_i v_i$. We may write $\lambda_i = |\lambda_i|e^{i\phi}$ where ϕ is as in the above part. Define $S(v_i) = |\lambda_i|v_i$ and $U(v_i) = e^{i\phi}v_i$, so that $T = SU$. Then $[S]_\beta$ is a diagonal matrix of real numbers, so that S is self-adjoint. We also have $[U]_\beta$ is a diagonal matrix of complex numbers on the unit circle, so that $[U]_\beta$ has orthonormal columns making U unitary.

3. Let T be an injective normal operator on $\mathbb{R}^4$ such that $T^* = T^2$ and that does not have 1 as an eigenvalue. Find the minimal polynomial m_T and the characteristic polynomial c_T of T.

Solution: Note that $T = (T^*)^* = (T^2)^* = (T^*)^2 = T^4$. This says that $p(x) = x^4 - x$ kills T, so m_T divides $x(x-1)(x^2+x+1)$, so m_T has possible factors $x, x-1, x^2+x+1$. Since T is injective, 0 is not an eigenvalue, and by assumption, T does not have 1 as an eigenvalue. The linear factors of m_T correspond to eigenvalues of T, so this forces $m_T(x) = x^2+x+1$. Since c_T and m_T have the same irreducible factors in $\mathbb{R}[x]$, this forces $c_T(x) = (x^2+x+1)^2$ because it has degree 4.

To see why c_T and m_T have the same irreducible factors, suppose that $c_T(x) = m_T(x)P(x)$ for some irreducible quadratic $P(x)$ with $P(x) \neq m_T(x)$. Over $\mathbb{C}$, $P(x)$ has some root, say r. Since roots of real quadratic polynomials come in conjugate pairs, the other root must be $\bar{r}$. As $m_T(x) \neq P(x)$, r is *not* a root of $m_T(x)$, because the coefficients of a polynomial are determined by its roots. But this then says that r is an eigenvalue of T (since it's a root of $c_T(x)$), and therefore *is* a root of m_T, a contradiction.

4. Consider the conic section $x^2 + 4y^2 + z^2 + 6xz - 4y - 4\sqrt{2}z = 1$ in $\mathbb{R}^3$. Find a rigid motion $f : \mathbb{R}^3 \to \mathbb{R}^3$ such that the conic takes the form $aX^2 + bY^2 + cZ^2 = 1$ for some $a, b, c \in \mathbb{R}$ when expressed in the variables $(X, Y, Z) = f(x, y, z)$.

Solution: Let β be the standard basis of $\mathbb{R}^3$ and let $v \in \mathbb{R}^3$ with $[v]_\beta = (x, y, z)$. We may rewrite the conic as $[v]_\beta^t A[v]_\beta + B[v]_\beta = 1$, where $A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 4 & 0 \\ 3 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -4 & -4\sqrt{2} \end{bmatrix}$. One can check that A has eigenvalues $\lambda = 4, 2$ with orthonormal bases of E_4 and E_2 given by $\{v_1, v_2\} = \{\{1/\sqrt{2}, 0, 1/\sqrt{2}), (0, 1, 0)\}$ and $\{v_3\} = \{(-1/\sqrt{2}, 0, 1/\sqrt{2})\}$ respectively. Then $\gamma = \{v_1, v_2, v_3\}$ is an eigenbasis of $\mathbb{R}^3$ for A. Let S_γ^β be the change of basis matrix from γ to β. By definition, we have $S_\gamma^\beta = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$, and $[v]_\beta = S_\gamma^\beta [v]_\gamma$. Letting $[v]_\gamma = (a, b, c)$, plugging this in we find $4a^2 + 4b^2 - 2c^2 - 4a - 4b - 4c = 1$. Completing the square yields $4(a-1/2)^2 + 4(b-1/2)^2 - 2(c+1)^2 = 1$. Setting $X = a - 1/2$, $Y = b - 1/2$ and $Z = c + 1$ gives $4X^2 + 4Y^2 - 2Z^2 = 1$. Since $[v]_\gamma = S_\beta^\gamma [v]_\beta$ and $S_\beta^\gamma = (S_\gamma^\beta)^{-1} = (S_\gamma^\beta)^t$, solving for (a, b, c) says the desired rigid motion is the map $(x, y, z) \to (x/\sqrt{2} + z/\sqrt{2} - 1/2, y - 1/2, -x/\sqrt{2} + z/\sqrt{2} + 1)$.