Selected Solutions to Homework 7

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7.2.6. Let A be an $n \times n$ matrix whose characteristic polynomial splits. Prove that A and A^t are similar.

Solution: Let J be a Jordan block of size $p \times p$ and let $\gamma = \{e_1, \ldots, e_p\}$ be the standard basis and set $\gamma' = \{e_p, \ldots, e_1\} = \{e'_1, \ldots, e'_p\}$. Then we have $Je'_i = Je_{p-i} = \lambda e_{p-i} + e_{p-i-1} = \lambda e'_i + e'_{i+1}$. It's then not too hard to see that $[J]_{\gamma'} = J^t$. This means that $S_\gamma^{\gamma'} J S_{\gamma'}^{\gamma} = [J]_{\gamma'} = S_\gamma^{\gamma'} J (S_\gamma^{\gamma'})^{-1} = [J]_{\gamma'} = J^t$, so that a Jordan block is similar to its transpose. Let $J = \mathrm{Diag}(J_1, \ldots, J_r)$ be the Jordan canonical form of A, and suppose that $S_i J_i S_i^{-1} = J_i^t$. This says that $S = \mathrm{Diag}(S_1, \ldots, S_r)$ conjugates J into J^t. Since $A = BJB^{-1}$ for some B and $SJS^{-1} = J^t$, we have $A^t = (B^t)^{-1} J^t B^t = (B^t)^{-1}(SJS^{-1})B^t = (S^{-1}B^t)^{-1} J (S^{-1}B^t)$. This says that A^t is similar to J and since A is similar to J, we deduce that A and A^t are similar and have the same Jordan form.

7.2.13. Let T be a nilpotent operator on an n-dimensional vector space V, and suppose that p is the smallest positive integer for which $T^p = 0$. Prove the following.

(b) There is a sequence of ordered bases $\beta_1, \beta_2, \ldots, \beta_p$ such that β_i is a basis for $\ker(T^i)$ and β_{i+1} contains β_i for $1 \le i \le p-1$.

(c) Let $\beta = \beta_p$ as above. Then $[T]_\beta$ is an upper triangular matrix with each diagonal entry equal to zero.

(d) The characteristic polynomial of T is x^n.

Solution:

(b) We have $\ker(T^i) \subset \ker(T^{i+1})$ for all i. Let β_1 be a basis of $\ker(T)$, and define β_2 by extending β_1 to a basis of $\ker(T^2)$. Inductively define β_{i+1} by extending the basis β_i of $\ker(T^i)$ to a basis of $\ker(T^{i+1})$.

(c) Note that the inclusion above is strict: suppose that $\ker(T^i) = \ker(T^{i+1})$ for some i. Then if $x \in \ker(T^j)$ for $j \ge i+1$, we have $T^j(x) = T^{i+1}(T^{j-i-1}(x)) = 0$. This says $T^{j-i-1}(x) \in \ker(T^{i+1}) = \ker(T^i)$, so that $T^{j-1}(x) = 0$ says $\ker(T^j) = \ker(T^{j-1})$. Applying this inductively shows that $\ker(T^i) = \ker(T^j)$ for all $j \ge i$. This says $V = \ker(T^p) = \ker(T^i)$ so that $T^i = 0$, a contradiction to the minimality of p. We may then write $\beta = \beta_1 \cup (\beta_2 \setminus \beta_1) \cup (\beta_3 \setminus \beta_2) \cup \ldots \cup (\beta_p \setminus \beta_{p-1})$ as a partition of β. Note that each set $\beta_i \setminus \beta_{i-1}$ consists of the basis vectors that are in $\ker(T^i)$ but not in $\ker(T^{i-1})$, i.e. they're the vectors we threw in to extend the basis. If $v \in \ker(T)$ then it corresponds to a zero column of $[T]_\beta$. Now for each $v \in \beta \setminus \beta_1$, we have $v \in \beta_i \setminus \beta_{i-1}$ for some unique i. Then $T(v) \in \ker(T^{i-1})$, so it can be written as a linear combination of vectors in $\ker(T^{i-1}) = \beta_1 \cup (\beta_2 \setminus \beta_1) \cup \ldots \cup (\beta_{i-1} \setminus \beta_{i-2})$. In particular, these vectors are all before v in our ordering of β, which says that $[T]_\beta$ is upper triangular as desired.

(d) The determinant of an upper triangular matrix is the product of its diagonal entries. In particular, this says that $c_T = \det(xI - [T]_\beta) = x^n$.

1. Let $A = \begin{pmatrix} 3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & -3 & 3 & 0 \\ 1 & 3 & -1 & 2 \end{pmatrix}$. Show that c_A splits, and find the Jordan canonical form B of A and an invertible matrix Q such that $B = Q^{-1}AQ$.

Solution: It's an easy computation to check that $c_T(x) = (x-2)^4$, so that c_T splits. To find the Jordan form of A, note that $A^3 = 0$ and $A^2 \neq 0$, so that $m_T(x) = x^3$. The constraints on the Jordan form say that the largest size Jordan block appears is a 3×3 block, which forces $J = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}$. Next, we find a Jordan canonical basis for A. We seek a cycle of length 1 and a cycle of length 3. A cycle of length 1 is merely an eigenvector. We have $A - 2I = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ -1 & -3 & 1 & 0 \\ 1 & 3 & -1 & 0 \end{pmatrix}$, so let's pick $v_1 = e_4$. To get a cycle of length 3, we seek a vector v_2 such that $(A-2I)^3 v_2 = 0$ but $(A-2I)^2 v_2 \neq 0$. Since $(A-2I)^3 = 0$ anyway, we just need a vector v_2 such that $(A-2I)^2 v_2 \neq 0$. Note that e_1 fits the bill, so a Jordan canonical basis is $\gamma = \{e_4, (A-2I)^2 e_1, (A-2I)e_1, e_1\} = \{(0,0,0,1), (1,-1,-2,2), (1,0,-1,1), (1,0,0,0)\}$. Take Q to be the change of basis matrix S_γ^β where β is the standard basis, and we have $J = Q^{-1}AQ$ as desired.

3. Find the Jordan canonical forms and minimal polynomials of the nilpotent matrices $A = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$.

Solution: Note by computation that $m_A(x) = x^3$ and $\dim(\ker(A)) = 3$. This says we have three Jordan blocks corresponding to $\lambda = 0$, and the largest sized block is 3, so we're forced to have two blocks of size 1. This says $J_A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$. Similarly, we find that $m_B(x) = x^4$, so that the Jordan form of B has a block of size 4. This forces a single remaining block of size 5, so that $J_B = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$.

4. Let $T : V \to V$ be a nilpotent linear transformation of F-vector spaces. Show that $\sum_{i=0}^{k} a_i T^i$ with $a_i \in F$ is nilpotent if and only if $a_0 = 0$.

Solution: Write $p_(T) = \sum_{i=0}^k a_i T^i$. First, suppose that $a_0 = 0$. Then we may write $p(T) = Tg(T)$ with $g(T) = \sum_{i=1}^k a_i T^{i-1}$. Note that T and $g(T)$ commute because they're both polynomials in T. Since T is nilpotent, we have $T^N = 0$ for some N and so $p(T)^N = T^N g(T)^N = 0$. This says that $p(T)$ is nilpotent. Conversely, suppose that $p(T)$ is nilpotent, say $p(T)^M = 0$ for some M. we have $a_0 I = p(T) + g(T)$ where $g(T) = a_0 I - p(T)$. Since $g(T)$ has no constant term, by what we just showed we must have that $g(T)^N = 0$. By the binomial theorem, we then have $a_0^{N+M} I = (p(T) + g(T))^{N+M} = \sum_{k=0}^{M+N} \binom{M+N}{k} p(T)^k g(T)^{N+M-k}$. For $0 \leq k \leq M$, $N + M - k \geq N$ so $g(T)^{N+M-k} = 0$. For $M \leq k \leq M + N$, we have $p(T)^k = 0$, so that $a_0^{M+N} I = 0$. This says $a_0^{N+M} = 0$, so that $a_0 = 0$ as desired.