Selected Solutions to Homework 6

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March 5, 2021

6.6.6. Let T be a normal operator on a finite-dimensional inner product space. Prove that if T is a projection, then T is also an orthogonal projection.

Solution: We know from previous homework that $\text{im}(T)^{\perp} = \ker(T^*)$. Since T is normal, from $3a$) here $\ker(T) = \ker(T^*)$. This says $\text{im}(T)^{\perp} = \ker(T)$ and $\text{im}(T) = \ker(T)^{\perp}$ since V is finite dimensional.

7.1.10 Let T be a linear operator on a finite-dimensional vector space whose characteristic polynomial splits, and let λ be an eigenvalue of T.

(a) Suppose that γ is a basis for K_{λ} consisting of the union of q disjoint cycles of generalized eigenvectors. Prove that $q \leq \dim(E_{\lambda})$.

(b) Let β be a Jordan canonical basis for T and suppose that $J = [T]_{\beta}$ has q Jordan blocks with λ in the diagonal positions. Prove that $q \leq \dim(E_{\lambda})$.

Solution:

(a) Each cycle in γ contains precisely one eigenvector. Therefore, γ contains q eigenvectors. Since these eigenvectors are part of a basis, they're linearly independent. Therefore, we can find at least q linearly independent eigenvectors, which tells us that $q \leq \dim(E_{\lambda})$ since the latter is the maximal number of linearly independent eigenvectors we can find.

(b) By definition of β, we have $\beta = \beta_1 \cup \ldots \cup \beta_k$ where β_i is a basis of K_{λ_i} that is a union of disjoint cycles. Since Jordan blocks correspond to cycles, the assumption of the problem says that the basis β_{λ} of K_{λ} breaks up into q disjoint cycles, so by the previous part we're done.

2. Let $T : V \to V$ be a normal operator on a finite dimensional inner product space. Show that the minimal polynomial of T is a product of distinct, monic, irreducible factors of degree 1 and 2.

Solution: Pick a basis β of V and let $[T]_{\beta} = A$. First, suppose that V is a complex inner product space. Then we know that A is diagonalizable over $\mathbb{C}$, so the minimal polynomial of A (which is the minimal polynomial of T) has distinct linear factors. Now suppose that V is a real inner product space, so that in particular, A is a real matrix. First, we show that the minimal polynomial of A (viewed over $\mathbb{R}$) is the same as the minimal polynomial of A (viewed over $\mathbb{C}$).

Let's denote the potentially different minimal polynomials by $m_{A,\mathbb{R}}$ and $m_{A,\mathbb{C}}$ respectively. Then since $m_{A,\mathbb{R}}$ has real coefficients (which are in particular, complex numbers) we have

$m_{A,\mathbb{C}} \mid m_{A,\mathbb{R}}$. Since both polynomials are *monic*, they're equal if they have the same degree. So suppose that $\deg(m_{A,\mathbb{C}}) < \deg(m_{A,\mathbb{R}})$. Then as A is a matrix with real entries, it's quite easy to see that $\overline{m_{A,\mathbb{C}}}(A) = 0$, where $\overline{m_{A,\mathbb{C}}}$ is the polynomial with conjugate coefficients to $m_{A,\mathbb{C}}$. Then $m_{A,\mathbb{C}} + \overline{m_{A,\mathbb{C}}}$ is a polynomial with real coefficients of that kills A, but $\deg(m_{A,\mathbb{C}} + \overline{m_{A,\mathbb{C}}}) < \deg(m_{A,\mathbb{R}})$, which is a contradiction to the definition of $m_{A,\mathbb{R}}$. Therefore, $m_{A,\mathbb{C}} = m_{A,\mathbb{R}}$ so it makes sense to speak of "the" minimal polynomial.

To finish up the problem, we now know that viewing A as a complex matrix, the minimal polynomial of A factors into a product of distinct linear factors in $\mathbb{C}$. Since the minimal polynomial is real, the complex roots come in conjugate pairs. Pairing them up yields an irreducible real quadratic factor. Doing this for all such factors over $\mathbb{C}$ tells us that m_A factors into a product of distinct linear factors (corresponding to real roots) and irreducible quadratic factors (corresponding to a pair of conjugate conplex roots) as desired.

3. Let $g : \mathbb{R}^n \to \mathbb{R}^n$ be a rigid motion, and let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a translation. Show that $g^{-1} \circ f \circ g$ is also a translation.

Solution: Since g is a rigid motion, we may write $g = h \circ T$ for some orthogonal operator T and translation h. Then $g^{-1} = T^{-1} \circ h^{-1}$, so $g^{-1} \circ f \circ g = T^{-1} \circ h^{-1} \circ f \circ h \circ T$. Note that $h^{-1} \circ f \circ h = f$, so $g^{-1} \circ f \circ g = T^{-1} \circ f \circ T$. For $v \in \mathbb{R}^n$, suppose that $f(v) = v + x$ for fixed x. Then $(T^{-1} \circ f \circ T)(v) = v + T^{-1}(x)$, so that $g^{-1} \circ f \circ g$ is a translation by $T^{-1}(x)$ as desired.

4. Determine the number of all possible Jordan canonical forms (up to ordering) of linear transformations with $c_T = (x+3)^8(x-2)^5(x-4)^2$ and $m_T(x) = (x+3)^5(x-2)^2(x-4)^2$.

Solution: We know that the algebraic multiplicity of an eigenvalue is equal to the sum of the sizes of the Jordan blocks for λ_i, and that the multiplicity of λ_i in the minimal polynomial is the size of the largest Jordan block that appears in the Jordan canonical form of T. Combining these two facts, we see the following:

$\lambda = -3$: We have a 5×5 block and the sum of sizes of blocks must be 8. The possibilities for the sizes of Jordan blocks (listed as tuples) are $(5,1,1,1)$, $(5,2,1)$, or $(5,3)$.

$\lambda = 2$: We have a 2×2 block and the sum of sizes of blocks must be 5. The possibilities for the sizes of Jordan blocks are $(2,2,1)$, $(2,1,1,1)$.

$\lambda = 4$: We have a 2×2 block and this is the *only* Jordan block.

This gives a total of 6 different Jordan forms (up to permutation of the factors).