Selected Solutions to Homework 5

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6.5.15 Let U be a unitary operator on an inner product space V, and let W be a finite-dimensional U-invariant subspace of V. Prove that:

(a) $U(W) = W$

(b) $W^\perp$ is U-invariant.

Solution:

(a) Since W is U-invariant, $U|_W$ is a linear operator on W. If $x \in \ker(U|_W)$, then because $x \in V$ we have $\|U|_W(x)\| = \|U(x)\| = \|x\| = 0$ as U is unitary. This says $U|_W$ is injective, and therefore surjective because W is finite dimensional.

(b) Let $x \in W^\perp$. We wish to show that $\langle U(x), y\rangle = 0$ for any $y \in W$. Since U is unitary, we have $\langle x, y\rangle = \langle U(x), U(y)\rangle = 0$. By part a), $U(W) = W$ so varying over all $y \in W$ means $U(y)$ varies over all of W. This says $U(x)$ is orthogonal to everything in W, so $U(x) \in W^\perp$ as desired.

6.5.31 Let H_u be a Householder operator on a finite-dimensional inner product space. Prove the following:

(c) Prove that $H_u(u) = -u$.

(d) Prove that $H_u^2 = I$ and $H_u^* = H_u$.

Solution:

(c) By definition, we have $H_u(u) = u - 2\langle u, u\rangle u = u - 2u = -u$ because u is a unit vector.

(d) By definition, we have $H_u(x) = x - 2\langle x, u\rangle u$. Then $H_u^2(x) = H_u(x - 2\langle x, u\rangle u) = H_u(x) - 2\langle x, u\rangle H_u(u) = (x - 2\langle x, u\rangle u) + 2\langle x, u\rangle u = x$, since $H_u(u) = -u$ by part c). This says $H_u^2(x) = x$, so $H_u^2 = I$ as desired.

Next, we wish to check that $\langle H_u(x), y\rangle = \langle x, H_u(y)\rangle$ for all $x, y \in V$. We have $\langle H_u(x), y\rangle = \langle x - 2\langle x, u\rangle u, y\rangle = \langle x, y\rangle - 2\langle x, u\rangle\langle u, y\rangle$. Similarly, we have $\langle x, H_u(y)\rangle = \langle x, y - 2\langle y, u\rangle u\rangle = \langle x, y\rangle + \langle x, -2\langle y, u\rangle u\rangle = \langle x, y\rangle - 2\overline{\langle y, u\rangle}\langle x, u\rangle$. Since $\overline{\langle y, u\rangle} = \langle u, y\rangle$, we're done.

6.8.24 Let T be a linear operator on a real inner product space V, and define $H : V \times V \to R$ by $H(x, y) = \langle x, T(y)\rangle$ for all $x, y \in V$.

(b) Prove that H is symmetric if and only if T is self-adjoint.

(c) What properties must T have for H to be an inner product on V?

(d) Explain why H may fail to be a bilinear form if V is a complex inner product space.

Solution:

(b) $H(x,y) = H(y,x)$ if and only if $\langle x, T(y)\rangle = \langle y, T(x)\rangle$ for all $x, y \in V$. Since V is a real inner product space, this says $\langle T(x), y\rangle = \langle x, T(y)\rangle$ for all $x, y \in V$, which is precisely what it means for T to be self-adjoint.

(c) An inner product must be bilinear, symmetric, and positive-definite: that it is, $\langle x, x\rangle > 0$ for $x \neq 0$. By part a), H is always bilinear, and by par b), H is symmetric if and only if T is self-adjoint. The positive definite condition says that T must be an operator with $\langle x, T(x)\rangle > 0$ for $x \neq 0$. So H defines an inner product if T is a self-adjoint operator with this property.

Note: A self-adjoint operator with this property is called *positive definite*. The point is that H inherits the properties of the operator T.

(d) H might fail to be a bilinear form over $\mathbb{C}$ because $H(x, iy) = \langle x, T(iy)\rangle = \langle x, iT(y)\rangle = -i\langle x, T(y)\rangle = -iH(x,y)$. In general, we do not necessarily have that $-iH(x,y) = iH(x,y)$. For example, take T to be the identity map. In this case, for non-zero x it's never true that $-i\|x\|^2 = i\|x\|^2$.

2. Let $H : V \times V \to F$ be a bilinear form on a finite-dimensional vector space V. Show that H is non-degenerate if and only if the only $w \in V$ such that $H(v,w) = 0$ for all $v \in V$ is $w = 0$.

Solution: Let L_H and R_H be the left and right functionals associated to H. Recall that H is non-degenerate if and only if R_H and L_H are both injective. Rephrased in this language, we wish to show that H is non-degenerate if and only if L_H is injective. The forward direction is free, so assume that L_H is injective. Then we wish to prove that R_H is also injective. In my discussion notes, I show that $(R_H)^*(\mathrm{eval}_v) = L_H(v)$ for any $v \in V$. The same argument can be used to show that $R_H(v) = (L_H)^*(\mathrm{eval}_v)$ for any $v \in V$. From HW 2, we know that $(L_H)^*$ is surjective because L_H is injective, which means that R_H is also surjective. Since V is finite dimensional, this means that R_H is injective as well, so we're done.

3. Consider the conic section $2x^2 - 72xy + 23y^2 - 140x + 20y - 75 = 0$ in $\mathbb{R}^2$. Perform a change of variables from (x,y) to (x', y') given by a rigid motion that transforms to conic to the form $\lambda(x')^2 + \mu(y')^2 = h$ for some $\lambda, \mu, h \in \mathbb{R}$. Use this to graph the original conic.

Solution: The idea is to perform a rotation to kill off the xy-term and then a translation to kill off the linear terms. First, we must rewrite our equation into something that linear algebra is equipped to handle. Let $v \in \mathbb{R}^2$ and let $[v]_\beta = (x,y)$, where β is the standard basis of $\mathbb{R}^2$. Rewriting the conic as a function of the vector v, we have the equation $[v]_\beta^t A[v]_\beta + B[v]_\beta = 75$, where $A = \begin{pmatrix} 2 & -36 \\ -36 & 23 \end{pmatrix}$ and $B = \begin{pmatrix} -140 & 20 \end{pmatrix}$. An orthonormal eigenbasis for the matrix A is given by $\gamma = \{(-3/5, 4/5), (4/5, 3/5)\}$ with associated eigenvalues 50 and -25 respectively. Letting S_γ^β denote the change of basis matrix from γ to β, we have $[v]_\beta = S_\gamma^\beta [v]_\gamma$ by definition. Substituting this in, we can rewrite our equation as $[v]_\gamma^t (S_\gamma^\beta)^t A S_\gamma^\beta [v]_\gamma + BS_\gamma^\beta [v]_\gamma = 75$. By definition, we have $S_\gamma^\beta = \begin{pmatrix} -3/5 & 4/5 \\ 4/5 & 3/5 \end{pmatrix}$. If we set $[v]_\gamma = (a,b)$, then in these new coordinates our equation is $50a^2 - 25b^2 + 100a - 100b = 75$. Completing the square yields $50(a+1)^2 -$

$25(b+2)^2 = 25$, so our equation is $2(a+1)^2 - (b+2)^2 = 1$. Letting $x' = a+1$ and $y' = b+2$ gives the equation $2(x')^2 - (y')^2 = 1$ in (x', y')-coordinates.