Selected Solutions to Homework 4

Tim Smits

February 12, 2021

6.2.14 Let W_1 and W_2 be subspaces of a finite-dimensional inner product space. Prove that $(W_1 + W_2)^\perp = W_1^\perp \cap W_2^\perp$ and $(W_1 \cap W_2)^\perp = W_1^\perp + W_2^\perp$.

Solution: Suppose that $x \in (W_1 + W_2)^\perp$. Then x is orthogonal to every vector in $W_1 + W_2$. In particular, $W_1, W_2 \subset W_1 + W_2$, so x is orthogonal to both W_1 and W_2, so $x \in W_1^\perp \cap W_2^\perp$. If $x \in W_1^\perp \cap W_2^\perp$, then x is orthogonal to every vector in both W_1 and W_2, so it's orthogonal to any sum of such vectors, so that $x \in (W_1 + W_2)^\perp$. This shows that $(W_1 + W_2)^\perp = W_1^\perp \cap W_2^\perp$. Now, since V is finite dimensional, $(W^\perp)^\perp = W$ for any subspace $W \subset V$. Replacing W_1 and W_2 with $W_1^\perp$ and $W_2^\perp$ in the above equality and taking an orthogonal complement then yields what we want.

6.3.12 Let V be an inner product space, and let T be a linear operator on V.

(a) Prove that $\operatorname{im}(T^*)^\perp = \ker(T)$.

(b) Prove that if V is finite dimensional, then $\operatorname{im}(T^*) = \ker(T)^\perp$.

Solution:

(a) Suppose that $x \in \ker(T)$. Then $\langle T(x), v\rangle = 0$ for any $v \in V$. Pulling the adjoint through the inner product, we have $\langle x, T^*(v)\rangle = 0$ for any $v \in V$, so that $x \in \operatorname{im}(T^*)^\perp$. The other containment is obvious since all these steps are easily reversible.

(b) Since V is finite dimensional, we have $(\operatorname{im}(T^*)^\perp)^\perp = \operatorname{im}(T^*)$, which tells us that $\operatorname{im}(T^*) = \ker(T)^\perp$.

6.4.8 Let T be a normal operator on a finite-dimensional complex inner product space V, and let W be a subspace of W. Prove that if W is T-invariant, then W is also T^*-invariant.

Solution: Since T is normal, it's diagonalizable, and we know that $T|_W$ is both well defined because W is T-invariant, and also that $T|_W$ is diagonalizable from previous homework. Let $\beta = \{v_1, \ldots, v_k\}$ be a basis of W consisting of eigenvectors of $T|_W$. If v_i has eigenvalue λ_i, then because T is normal we know that it's an eigenvector for T^* of eigenvalue $\overline{\lambda_i}$. Thus, $T^*(\beta) \subset W$ so that W is T^*-invariant as desired.

6.4.14 Let V be a finite dimensional real inner product space, and let U, T be self-adjoint linear operators on V with $UT = TU$. Prove that there is an orthonormal basis of V consisting of eigenvectors of both U and T.

Solution: Since U, T are self-adjoint operators, they're diagonalizable. Let E_λ be an eigenspace of T. On the previous homework, we showed that we can find a basis β_λ for E_λ consisting of eigenvectors for both U and T. Running Gram-Schmidt on this eigenbasis turns it into an *orthonormal* eigenbasis β'_λ, because the process of Gram-Schmidt just replaces the basis vectors with specific linear combinations of said basis vectors, and linear combinations of eigenvectors clearly remain eigenvectors. Let $\lambda_1, \ldots, \lambda_k$ be the distinct eigenvalues of T. The eigenspaces E_{λ_i} are all orthogonal to each other because T and U are self-adjoint (so in particular, they're normal). Then as before, taking the union $\beta' = \beta'_{\lambda_1} \cup \ldots \cup \beta'_{\lambda_k}$ then provides an orthonormal basis for $E_{\lambda_1} \oplus \ldots \oplus E_{\lambda_k} = V$ of eigenvectors for both U and T.

2. Let T be a self-adjoint operator on a finite-dimensional inner product space.

(a) Show that $\|T(v) \pm iv\|^2 = \|T(v)\|^2 + \|v\|^2$.

(b) Show that $T \pm i$ is invertible.

Solution:

(a) By definition, we have $\|T(v) \pm iv\|^2 = \langle T(v) \pm iv, T(v) \pm iv\rangle = \langle T(v), T(v)\rangle \pm \langle T(v), iv\rangle \pm \langle iv, T(v)\rangle + \langle iv, iv\rangle$. The middle two terms cancel by anti-linearity of the inner product and the self-adjointness of T. As $\langle iv, iv\rangle = i(-i)\langle v, v\rangle$, we're just left with $\langle T(v), T(v)\rangle + \langle v, v\rangle = \|T(v)\|^2 + \|v\|^2$.

(b) Suppose that $x \in \ker(T \pm i)$. Then part a) says that $\|(T \pm i)(x)\|^2 = \|T(x)\|^2 + \|x\|^2 = 0$, which in particular, forces $\|x\|^2 = 0$. This says $x = 0$, so $T \pm i$ is injective. Since it's a linear operator on a finite dimensional vector space, it's therefore an isomorphism, so invertible.