Selected Solutions to Homework 3

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5.4.6 For each linear operator T on the vector space V, find an ordered basis for the T-cyclic subspace generated by the vector z.

(a) $V = \mathbb{R}^4$, $T(a,b,c,d) = (a+b, b-c, a+c, a+d)$, $z = e_1$.

(d) $V = M_2(\mathbb{R})$, $T(A) = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} A$, $z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Solution:

(a) The T-cyclic subspace spanned by z is by definition $W = \operatorname{Span}\{z, T(z), T^2(z), \ldots\}$. We have $T(z) = (1,0,1,1)$, $T^2(z) = (1,-1,2,2)$, and $T^3(z) = (0,-3,3,3)$. Notice that $T^3(z) = (3T^2 - 3T)(z)$, so that $T^i(z) \in \operatorname{Span}\{T(z), T^2(z)\}$ for $i \geq 3$. This says $W = \operatorname{Span}\{z, T(z), T^2(z)\}$. Since $T(z)$ and $T^2(z)$ are not multiples of each other, $\{T(z), T^2(z)\}$ is linearly independent and by inspection we see that $e_1 \notin \operatorname{Span}\{T(z), T^2(z)\}$, so that $\{z, T(z), T^2(z)\}$ is a basis for W.

(d) The T-cyclic subspace spanned by z is by definition $W = \operatorname{Span}\{z, T(z), T^2(z), \ldots\}$. Note that $T^2(z) = 3T(z)$, so $W = \operatorname{Span}\{z, T(z)\}$. It's clear that z and $T(z)$ are not multiples of each other, so $\{z, T(z)\}$ is a basis of W.

5.4.20) Let T be a linear operator on a vector space V, and suppose that V is a T-cyclic subspace of itself. Prove that if U is a linear operator on V, then $UT = TU$ if and only if $U = g(T)$ for some $g(t) \in F[t]$.

Solution: Since V is T-cyclic, we have a basis given by $\beta = \{v, T(v), \ldots, \}$ for some v. For any $v \in V$, we have $U(v) \in V$ so we can write $U(v) = c_1 v + c_2 T(v) + \ldots + c_n T^{n-1}(v) = (c_1 \mathrm{id}_V + \ldots + c_n T^{n-1})(v)$ for some $c_1, \ldots, c_n \in F$ and some n, so $U(v) = g(T)(v)$ where $g(x) = c_1 + \ldots + c_n x^{n-1}$.

Suppose that $UT = TU$. Then it's a simple induction argument to see that for any $i \geq 0$, we have $UT^i = T^i U$, and so $U(T^i(v)) = T^i(U(v)) = T^i(g(T)(v)) = g(T)(T^i(v))$ (because powers of T commute), so that $U = g(T)$ because they agree on a basis. Conversely, let $U = g(T)$ for some $g(x) \in F[x]$. For any $i \geq 0$, We have $(UT)(T^i(v)) = U(T^{i+1}(v)) = g(T)(T^{i+1}(v)) = T(g(T)(T^i(v)) = T(U(T^i(v))) = (TU)(T^i(v))$. This says UT and TU agree on a basis, so $UT = TU$ as desired.

5.2.19(a) Show that if T and U are simultaneously diagonalizable operators, then T and U commute.

Solution: Let $\beta = \{v_1, \ldots, v_n\}$ be a simultaneous eigenbasis for U and T. For each v_i, let λ_i be the associated eigenvalue for T, and γ_i the associated eigenvalue for U. We have $(UT)(v_i) = U(T(v_i)) = U(\lambda_i v_i) = \lambda_i(U(v_i)) = \lambda_i \gamma_i v_i$, and $(TU)(v_i) = T(\gamma_i v_i) = \gamma_i T(v_i) = \gamma_i \lambda_i v_i$. Since UT and TU agree on a basis, they're equal as operators.

5.4.25(a) Prove the converse to 5.2.19(a): if T and U are diagonalizable linear operators on a finite dimensional vector space V with $UT = TU$, then T and U are simultaneously diagonalizable.

Solution: Suppose that $UT = TU$. Let λ be an eigenvalue of T, with associated eigenspace E_λ. I claim that E_λ is U-invariant: indeed, for $v \in E_\lambda$, we have $T(U(v)) = (TU)(v) = (UT)(v) = U(T(v)) = U(\lambda v) = \lambda U(v)$. This says that $U(v)$ is an eigenvector of T, so that $U(v) \in E_\lambda$, i.e. E_λ is U-invariant. Then $U|_{E_\lambda}$ is a well defined linear operator, and by 5.4.24, we know that $U|_{E_\lambda}$ is diagonalizable. Therefore, we can find a basis $\beta_\lambda \subset E_\lambda$ of eigenvectors for $U|_{E_\lambda}$. Since the vectors in β_λ still live in E_λ, in particular, they're still eigenvectors of T, so β_λ is a basis of E_λ consisting of eigenvectors for both T and $U|_{E_\lambda}$. As T is diagonalizable, we have $V = E_{\lambda_1} \oplus \ldots \oplus E_{\lambda_k}$ where $\lambda_1, \ldots, \lambda_k$ are the eigenvalues of T, and what we know about direct sums says that $\beta = \beta_{\lambda_1} \cup \ldots \cup \beta_{\lambda_k}$ is a basis for V. If $v \in \beta_{\lambda_i}$, then $U(v) = U|_{E_{\lambda_i}}(v)$, so this says that v is an eigenvector for U as well. Therefore, β is a basis of V consisting of eigenvectors for both U and T, so we're done.

2 Let V be a finite dimensional F-vector space, and let $T : V \to V$ be a linear operator with $T^3 = T$.

(a) Show that if $F = \mathbb{R}$, then T is diagonalizable.

(b) Give an example to show that if $F = \mathbb{F}_2$, then T need not be diagonalizable.

Solution:

(a) Let $m(x)$ be the minimal polynomial of T. Since $T^3 = T$, this says $x^3 - x$ kills T, so that $m(x) \mid x^3 - x$. Since $x^3 - x = x(x-1)(x+1)$ is a product of distinct linear factors in $\mathbb{R}[x]$, this says that $m(x)$ must also split, so that T is diagonalizable.

(b) The companion matrix C_p for $p(x) = x^3 + x \in \mathbb{F}_2[x]$ over $\mathbb{F}_2$ is given by $\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, and from discussion we know that C_p has characteristic polynomial $x^3 + x$. Therefore, the linear operator $T : v \to C_p v$ on $\mathbb{F}_2^3$ satisfies the polynomial equation $T^3 = T$ by the Cayley-Hamilton theorem. Since C_p has rank 2, we have $\dim(E_0) = \dim(\ker(C_p)) = 1$, while $\dim(E_1) = \dim(\ker(C_p - I_3)) = 1$ as well. This says we can't find an eigenbasis for T, so that T is not diagonalizable.