Selected Solutions to Homework 2

Tim Smits

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2.6.12 Let V be a finite dimensional vector space with ordered basis β. Prove that $\psi(\beta) = \beta^{**}$ where $\psi : V \to V^{**}$ is the map $\psi(v) = \mathrm{ev}_v$.

Solution: Let $\beta = \{v_1, \dots, v_n\}$. Then the dual basis to β is given by $\{f_1, \dots, f_n\}$ where $f_i(v_j) = 1$ if $i = j$ and 0 otherwise. The dual basis β^{**} to β^* is given by $\{\delta_1, \dots, \delta_n\}$ where $\delta_i(f_j) = 1$ if $i = j$ and 0 otherwise. For any v_i, we have $\psi(v_i) = \mathrm{ev}_{v_i}$, and I claim that $\psi(v_i) = \delta_i$. To do so, it's sufficient to check that both sides agree on the basis β^*. We have $\psi(v_i)(f_j) = \mathrm{ev}_{v_i}(f_j) = f_j(v_i) = 1$ if $i = j$ and 0 otherwise. This is precisely the definition of δ_i, so we're done.

5.1.18 Let $T : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ be defined by $T(A) = A^t$.

(a) Show that ± 1 are the only eigenvalues of T.

(b) Describe the eigenvectors corresponding to each eigenvalue of T.

(d) Find an ordered basis β of $M_n(\mathbb{R})$ for $n > 2$ such that $[T]_\beta$ is diagonal.

Solution:

(a) Let $A = (a_{ij})$. If λ is an eigenvalue of T, then $A^t = \lambda A$. The ij-th entry of A^t is a_{ji}, so we have $a_{ji} = \lambda a_{ij}$. Similarly, the ji-th entry of A^t is a_{ij}, so $a_{ij} = \lambda a_{ji}$. This says that $a_{ij} = \lambda^2 a_{ij}$, so that $\lambda^2 = 1$ yields $\lambda = \pm 1$ as desired.

(b) An eigenvector of eigenvalue 1 is a matrix A with $A = A^t$, i.e. a symmetric matrix. An eigenvector of eigenvalue 1 is a matrix A with $A = -A^t$, i.e. a skew-symmetric matrix.

(d) I claim that $M_n(\mathbb{R}) = \mathrm{Sym}_n(\mathbb{R}) \oplus \mathrm{Skew}_n(\mathbb{R})$. If β is a basis of $\mathrm{Sym}_n(\mathbb{R})$ and γ is a basis of $\mathrm{Skew}_n(\mathbb{R})$, then we know that $\beta \cup \gamma$ is a basis of $\mathrm{Sym}_n(\mathbb{R}) \oplus \mathrm{Skew}_n(\mathbb{R})$. Since each vector in $\beta \cup \gamma$ is an eigenvector of T, we will then have that $[T]_{\beta\cup\gamma}$ is diagonal.

Any matrix A in $M_n(\mathbb{R})$ can be written as $A = \frac{1}{2}(A + A^t) + \frac{1}{2}(A - A^t)$. Since the transpose map is a linear involution, we see that the first matrix is symmetric while the second is skew symmetric, so $M_n(\mathbb{R}) = \mathrm{Sym}_n(\mathbb{R}) + \mathrm{Skew}_n(\mathbb{R})$. If $A \in \mathrm{Sym}_n(\mathbb{R}) \cap \mathrm{Skew}_n(\mathbb{R})$, then $A = A^t$ and $A = -A^t$, so $2A^t = 0$ says $A = A^t = 0$. Therefore, $M_n(\mathbb{R}) = \mathrm{Sym}_n(\mathbb{R}) \oplus \mathrm{Skew}_n(\mathbb{R})$. It remains to give an explicit description of the bases β and γ.

I claim that $\beta = \{E_{ij} + E_{ji} \ : 1 \le i \le j \le n\} = \{F_{ij}\}$ is a basis for $\mathrm{Sym}_n(\mathbb{R})$, where E_{ij} is the matrix with ij-th entry 1 and 0 everywhere else. Because none of these matrices share any non-zero entries in common, they must be linearly independent. If $A = (a_{ij})$ is a symmetric matrix, then $a_{ij} = a_{ji}$, so we need only specify the entries on or below the diagonal of A. It's then quite clear that $A = \sum_{i=1}^n \frac{1}{2} a_{ii} F_{ii} + \sum_{j<i} a_{ij} F_{ij}$ so it's a basis.

Similarly, one can show that $\gamma = \{E_{ij} - E_{ji} : 1 \leq i < j \leq n\}$ is a basis for $\mathrm{Skew}_n(\mathbb{R})$. Therefore, $[T]_{\beta \cup \gamma}$ is diagonal with $\frac{n(n+1)}{2}$ 1's and $\frac{n(n-1)}{2}$ -1's along the diagonal.

2. Let V be an infinite dimensional vector space with basis $B = \{v_i : i \in I\}$ for some indexing set I. Let f_i be the dual vector to v_i, i.e. $f_i(v_j) = 1$ if $i = j$ and 0 otherwise. Show that $B^* = \{f_i : i \in I\}$ does not span V^*.

Solution: Let $f \in \mathrm{Span}(B^*)$, so that $f = c_1 f_{i_1} + \ldots + c_k f_{i_k}$ for some $i_1, \ldots, i_k \in I$ and $c_i \in F$. Then the definition of the f_i means that f can only take on finitely many non-zero values on B, namely $f(v_j) = c_j$ for $j \in \{i_1, \ldots, i_k\}$ and is 0 otherwise. As we've seen, any map of sets $f : B \to F$ extends to a functional $T_f : V \to F$. Consider the map $f : B \to F$ defined by $f(v_i) = 1$ for all $i \in I$. Then the functional T_f satisfies $T_f(v_i) = f(v_i) = 1$ for all $i \in I$, so the above discussion shows that $T_f \notin \mathrm{Span}(B^*)$.

4. Let C^∞ be the $\mathbb{R}$-vector space of smooth functions $f : \mathbb{R} \to \mathbb{R}$. Consider the linear operator $T = \frac{d^2}{dx^2} : C^\infty \to C^\infty$. For each eigenvalue of T, find two linearly independent eigenvectors.

Solution: Let λ be an eigenvalue of T. We consider the three cases $\lambda = 0, \lambda > 0$, and $\lambda < 0$.

$\underline{\lambda = 0}$: In this case, we are looking for functions f such that $f'' = 0$. Integrating twice says $f(x) = c_1 + c_2 x$ for some $c_1, c_2 \in \mathbb{R}$. Then clearly $\{1, x\}$ will be a linearly independent set of eigenvectors: if $c_1 + c_2 x = 0$, then plugging in $x = 0$ says $c_1 = 0$, so that $c_2 = 0$.

$\underline{\lambda > 0}$: We seek functions f such that $f'' = \lambda f$. Note that $f = e^{\sqrt{\lambda}x}$ and $e^{-\sqrt{\lambda}x}$ both work, so we just need to show $\{e^{\sqrt{\lambda}x}, e^{-\sqrt{\lambda}x}\}$ is linearly independent. Suppose that $c_1 e^{\sqrt{\lambda}x} + c_2 e^{-\sqrt{\lambda}x} = 0$ for some $c_1, c_2 \in \mathbb{R}$. Plugging in $x = 0$ says $c_1 + c_2 = 0$ and differentiating and plugging in $x = 0$ says $c_1\sqrt{\lambda} - c_2\sqrt{\lambda} = 0$. Since $\lambda \neq 0$, dividing the second equation by $\sqrt{\lambda}$ easily shows us that $c_1 = c_2 = 0$.

$\underline{\lambda < 0}$: Set $\lambda = -\alpha$, so we seek functions f such that $f'' = -\alpha f$. Note that $f = \sin(\sqrt{\alpha}x)$ and $f = \cos(\sqrt{\alpha}x)$ both work, so we just need to show that $\{\sin(\sqrt{\alpha}x), \cos(\sqrt{\alpha}x)\}$ are linearly independent. Suppose that $c_1 \sin(\sqrt{\alpha}x) + c_2 \cos(\sqrt{\alpha}x) = 0$ for some $c_1, c_2 \in \mathbb{R}$. Plugging in $x = 0$ gives $c_2 = 0$ and plugging in $x = \frac{\pi}{2\sqrt{\alpha}}$ gives $c_1 = 0$, so we're done.