# Selected Solutions to Homework 1

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**2.** Let $V = \mathbb{F}_2^n$, viewed as an $\mathbb{F}_2$ vector space.

(a) Show that $V$ has $2^n - 1$ different one-dimensional subspaces.

(b) Find the number of 2-dimensional subspaces of $V$.

**Solution:**

(a) By definition, any one-dimensional subspace looks like $W = \text{Span}\{v\}$ for some non-zero $v \in V$. Explicitly, we have $\text{Span}\{v\} = \{0, v\}$ because we are working over $\mathbb{F}_2$, so there are only two scalars to multiply by. Therefore, we see that distinct non-zero vectors correspond to distinct one-dimensional subspaces. Since there are $2^n - 1$ non-zero vectors in $V$, we get $2^n - 1$ one-dimensional subspaces.

(b) By definition, any 2-dimensional subspace looks like $W = \text{Span}\{v_1, v_2\}$ for $v_1, v_2$ linearly independent vectors in $V$. Saying $\{v_1, v_2\}$ is linearly independent is the same as saying that $v_2$ is not an $\mathbb{F}_2$-multiple of $v_1$, i.e. $v_2 \neq v_1$ because the only scalars in $\mathbb{F}_2$ are 0 and 1. Therefore, any two choices of distinct non-zero vectors gives a linearly independent subset, for a total of $\binom{2^n-1}{2}$ possible linearly independent subsets of two elements. For each subspace $W = \text{Span}\{v_1, v_2\}$ there are $\binom{3}{2}$ different possible bases for $W$. This is because we have $W = \{0, v_1, v_2, v_1 + v_2\}$, and a choice of any two distinct elements among the three non-zero vectors produces a linearly independent subset of two elements, i.e. a basis of $W$. This gives a total of $\frac{\binom{2^n-1}{2}}{\binom{3}{2}}$ different 2-dimensional subspaces.

**3.** View $\mathbb{R}$ as a $\mathbb{Q}$-vector space. Prove that $\mathbb{R}$ has an uncountable basis.

**Solution:** Suppose that $\mathbb{R}$ was a finite dimensional $\mathbb{Q}$-vector space. Then we would have $\mathbb{R} \cong \mathbb{Q}^n$ for some $n \geq 1$. Since $\mathbb{Q}$ is countable, $\mathbb{Q}^n$ is countable because it's a finite product of countable sets, and because $\mathbb{R}$ is uncountable this leads to a contradiction. Therefore, $\mathbb{R}$ is infinite dimensional. Suppose that $\beta = \{v_i\}_{i \in \mathbb{N}}$ is a countable basis for $\mathbb{R}$ as a $\mathbb{Q}$-vector space. Since every element of $\mathbb{R}$ is a *finite* $\mathbb{Q}$-linear combination of elements of $\beta$, we see that $\mathbb{R} = \bigcup_{n=1}^{\infty} \text{Span}\{v_1, v_2, \ldots, v_n\}$. As $\text{Span}\{v_1, \ldots, v_n\} \cong \mathbb{Q}^n$ as $\mathbb{Q}$-vector spaces (they're both $n$-dimensional), the right hand side is a countable union of countable sets, and therefore countable, which gives a contradiction. Thus, $\mathbb{R}$ has uncountable dimension over $\mathbb{Q}$.

**7.** Let $V$ be a finite dimensional $F$-vector space, and let $T : V \to V$ be a linear transformation.

(a) Show that there exists $k \geq 1$ such that $\ker(T^k) = \ker(T^m)$ for all $m \geq k$.

(b) Use part $(a)$ to show that there exists $k \geq 1$ such that $\ker(T^k) \cap \text{im}(T^k) = 0$.

**Solution:**

(a) Note that for any $k \geq 0$ we have $\ker(T^k) \subset \ker(T^{k+1})$, because if $T^k(x) = 0$ for some $x \in V$, we have $T^{k+1}(x) = T(T^k(x)) = T(0) = 0$. Now suppose to the contrary, that for all $k \geq 1$, there exists an integer $m \geq k$ such that $\ker(T^k) \neq \ker(T^m)$, i.e. we have a strict containment $\ker(T^k) \subset \ker(T^m)$. Let $m_1$ be such a choice of $m$ for $k = 1$, and define $m_i$ for $i \geq 2$ by such a choice of $m$ for $k = m_{i-1}$. Then we are able to construct a strictly increasing sequence of subspaces $0 \subset \ker(T) \subset \ker(T^{m_1}) \subset \ker(T^{m_2}) \subset \ldots$. Since the containment at each stage is strict, we must have $\dim\ker(T^{m_i}) > \dim\ker(T^{m_{i-1}})$, and since $V$ is finite dimensional, there must eventually be some stage $j$ where $\dim\ker(T^{m_j}) = \dim(V)$, i.e. $\ker(T^{m_j}) = V$, so that $T^{m_j} = 0$. However, we then have for $m \geq m_j$, that $T^m = T^{m_j} = 0$, contradicting our assumption. This proves what we want to show.

(b) Let $k$ be as in part $(a)$, and let $x \in \ker(T^k) \cap \mathrm{im}(T^k)$. Then $T^k(x) = 0$ and $x = T^k(y)$ for some $y \in V$. Applying $T^k$ to both sides says $0 = T^k(x) = T^{2k}(y)$, so that $y \in \ker(T^{2k})$. Since $\ker(T^k) = \ker(T^{2k})$ by assumption, this says $x = 0$ as desired.

**8.** Let $T : \mathbb{F}_2^3 \to \mathbb{F}_2^3$ denote the linear transformation that is represented by the matrix $[T]_\beta = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$, with respect to $\beta = \{(0,1,1),(1,1,0),(0,0,1)\} = \{v_1, v_2, v_3\}$. Find $[T]_\gamma$, where $\gamma = \{e_1, e_2, e_3\}$ is the standard basis of $\mathbb{F}_2^3$.

**Solution:** By definition, we have $[T]_\gamma = \begin{bmatrix} | & | & | \\ [T(e_1)]_\gamma & [T(e_2)]_\gamma & [T(e_3)]_\gamma \\ | & | & | \end{bmatrix}$, where $[T(e_i)]_\gamma$ denotes the $\gamma$-coordinates of $T(e_i)$. By inspection, we have $e_1 = v_1 + v_2 + v_3$, $e_2 = v_1 + v_3$, and $e_3 = v_3$. Therefore, $[T(e_1)]_\beta = [T]_\beta[e_1]_\beta = (0,1,1)$, $[T(e_2)]_\beta = [T]_\beta[e_2]_\beta = (1,0,0)$ and $[T(e_3)]_\beta = [T]_\beta[e_3]_\beta = (0,0,1)$. Changing to $\gamma$-coordinates, we have $[T(e_1)]_\gamma = v_2 + v_3 = (1,1,1)$, $[T(e_2)]_\gamma = v_1 = (0,1,1)$ and $[T(e_3)]_\gamma = (0,0,1)$, so that $[T]_\gamma = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$.