Final Review Problems

1. Let $A = \begin{pmatrix} 1 & 0 & 2 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 3 & -2 \\ 0 & 2 & 2 & -1 \end{pmatrix}$

   (a) Let J be a Jordan block of size n and eigenvalue λ. Compute the rational canonical form of J.

   (b) Compute the Jordan canonical form of A.

   (c) Compute the rational canonical form of A.

2. Give an example of an operator $T : \mathbb{R}^6 \to \mathbb{R}^6$ such that $T^4 + T^2 + I = 0$. How many possible conjugacy classes of operators are there with this property?

3. Let $J \subset M_n(\mathbb{C})$ be a Jordan block of size n corresponding to λ.

   (a) Prove that the Jordan form of J^2 is the Jordan block of size n corresponding to λ^2 if $\lambda \neq 0$.

   (b) Prove that if $\lambda = 0$ then the Jordan form of J^2 has two blocks (both corresponding to 0) of size $\frac{n}{2}, \frac{n}{2}$ if n is even or $\frac{n-1}{2}, \frac{n+1}{2}$ if n is odd.

   (c) For each $n \geq 2$, give an example of a matrix $A \in M_n(\mathbb{C})$ that has no square root.

4. Let $A \in M_n(\mathbb{C})$. Prove that A is nilpotent if and only if $\text{Tr}(A^k) = 0$ for all $k \geq 1$.

5. (If you've taken analysis) View $M_n(\mathbb{C})$ as a metric space by identifying it with $\mathbb{C}^{n^2}$. Show that the set of diagonalizable matrices is *dense* in $M_n(\mathbb{C})$. That is, show that for any $A \in M_n(\mathbb{C})$ there is a sequence of diagonalizable matrices D_n such that $\lim_{n\to\infty} D_n = A$.

Solutions

1. I intended for this to use the invariant factor rational canonical form, but I'll do it with the elementary divisor rational canonical form for the sake of the example. The intended solution was to use part (a) to conjugate the Jordan canonical form to the invariant factor form of the rational canonical form.

   (a) The minimal and characteristic polynomial of J are both given by $(x-\lambda)^n$. The elementary divisors corresponding to $(x-\lambda)$ have the constraint that their product is $(x-\lambda)^n$ and their least common multiple gives back $(x-\lambda)^n$ as well. In particular, this says we can only have the single elementary divisor $(x-\lambda)^n$, so that the rational canonical form of J is the companion matrix of $(x-\lambda)^n$.

   (b) We find that $c_A(x) = (x-1)^4$, and that $m_A(x) = (x-1)^2$ by computation. This says that the largest block size corresponding to 1 is 2. One can also check that $\ker(A - I)$ is two dimensional, so we have two Jordan blocks. This says that the Jordan form of A is given by $J = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

(c) The computation of the elementary divisor rational canonical form is essentially identical to the computation of the Jordan canonical form. In particular, the constraints on the elementary divisors say the largest power of $x-1$ that can appear our list is 2, so the possible lists are $\{x-1, x-1, (x-1)^2\}$ and $\{(x-1)^2, (x-1)^2\}$. We're interested in $E_{x-1} = \ker(A-I)$, which we already have found to be two dimensional. Therefore, there are $2/1 = 2$ elementary divisors corresponding to $x-1$, so our list is $\{(x-1)^2, (x-1)^2\}$.

This says the elementary divisor rational canonical form is $\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{pmatrix}$.

2. Let C_f denote the companion matrix of the polynomial $f(x)$. We know that the minimal polynomial and characteristic polynomial of C_f are equal, and given by f. Recall that the minimal polynomial of a block diagonal matrix is given by the least common multiple of the minimal polynomials of each of its blocks. With this in mind, we may take $A = \text{Diag}(0, 0, C_{x^4+x^2+1})$ and $T(x) = Ax$ as an example of such an operator.

To classify all such operators up to conjugacy, we may count the number of possible rational canonical forms (either version) of T. Note that $x^4+x^2+1 = (x^2-x+1)(x^2+x+1)$, so there are three possibilities for $m_T : x^2-x+1, x^2+x+1, x^4+x^2+1$. Since the minimal polynomial and the characteristic polynomial have the same irreducible factors and $\deg(c_T) = 6$, we must have that $c_T = (x^2-x+1)^2(x^2+x+1)$ or $c_T = (x^2-x+1)(x^2+x+1)^2$. For your sake, I'll use the elementary divisor version of the rational canonical form.

First, suppose that $c_T = (x^2-x+1)^2(x^2+x+1)$. The product of the elementary divisors must give c_T, so the possible lists are $\{x^2-x+1, x^2-x+1, x^2+x+1\}$ or $\{(x^2-x+1)^2, x^2+x+1\}$. Since the least common multiple of the elementary divisors must give back the minimal polynomial and the minimal polynomial has no repeated irreducible factors, the second list is not possible so there is a single choice of rational canonical form.

Similarly, suppose that $c_T = (x^2-x+1)(x^2+x+1)^2$. The product of the elementary divisors must give back c_T, so the possible lists are $\{x^2-x+1, x^2+x+1, x^2+x+1\}$ and $\{x^2-x+1, (x^2+x+1)^2\}$. The latter is again not possible because the minimal polynomial must have no repeated irreducible factors, so once again there's a single rational canonical form.

Combing the two cases, we end up with a total of two different conjugacy classes of operators.

3. (a) First, observe that the only eigenvalue of J^2 is λ^2: this is because J is upper triangular so J^2 has λ^2 as its diagonal entries. Note that $J^2 - \lambda^2 I = (J+\lambda I)(J-\lambda I)$. The matrix $J+\lambda I$ has a single eigenvalue, namely $2\lambda \neq 0$. This says that $J+\lambda I$ is invertible, so that $\ker(J^2-\lambda^2 I) = \ker(J-\lambda I)$. Since the dimension of $\ker(J^2-\lambda^2 I)$ is the number of Jordan blocks corresponding to λ^2 in the Jordan form of J^2, we see that the Jordan form of J^2 consists of a single Jordan block corresponding to the eigenvalue λ.

(b) First suppose that n is even. Since the minimal polynomial of J is given by x^n, we easily see that the minimal polynomial of J^2 is $x^{n/2}$. The power of x in the minimal polynomial tells us the size of the largest Jordan block corresponding to 0, which must be $n/2$. Next, it's easy to check that $\dim\ker(J^2) = 2$ by computation, so we have two Jordan blocks for 0. This forces the Jordan form of J to have two blocks corresponding to 0 of size $n/2$. Similarly, if n is odd then the minimal polynomial of J^2 is given by $x^{(n+1)/2}$. The argument above shows that we have exactly two Jordan blocks corresponding to 0, and therefore they must have sizes $(n-1)/2$ and $(n+1)/2$ as desired.

(c) Let J be a Jordan block of size n corresponding to 0. I claim that J has no square root. Suppose otherwise, that $J = S^2$ for some matrix S. As $J^n = 0$, we must have that $S^{2n} = 0$. This says that the minimal polynomial of S is a power of x, and since the minimal and characteristic polynomial of S have the same irreducible factors, we actually see that $S^n = 0$. If n is even, we see that $J^{n/2} = (S^2)^{n/2} = S^n = 0$ and if n is odd, we see that $J^{(n+1)/2} = (S^2)^{(n+1)/2} = S^{n+1} = 0$. In either case, we have $n/2 < n$ and $(n+1)/2 < n$, which contradict that fact that the minimal polynomial of J is x^n. Therefore, J has no square root as desired.

4. Firstly, observe that if $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A, then $\lambda_1^k, \ldots, \lambda_n^k$ are the eigenvaues of A. Let J be the Jordan form of A. Then A and J are similar, so A^k and J^k are also similar. J is an upper triangular matrix with λ_i on it's diagonals, and so J^k has λ_i^k as its diagonals. It's then clear from the definition of the characteristic polynomial that A^k has eigenvalues $\lambda_1^k, \ldots, \lambda_n^k$.
Now, suppose that A is nilpotent. Since the trace of a matrix is the sum of its eigenvalues and the eigenvalues of A^k for any k are still just 0, we see that $\mathrm{Tr}(A^k) = 0$ for all $k \geq 1$. Conversely, suppose that $\mathrm{Tr}(A^k) = 0$ for all $k \geq 1$. For contradiction, suppose that A has non-zero eigenvalues $\lambda_1, \ldots, \lambda_r$ of multiplicities $m_1, \ldots, m_r$. Then By our earlier observation, we have $\mathrm{Tr}(A^k) = m_1\lambda_1^k + \ldots + m_r\lambda_r^k = 0$. Writing down this relation for $k = 1, 2, \ldots, r$ gives a matrix equation
$$\begin{pmatrix} \lambda_1 & \lambda_2 \ldots & \lambda_r \\ \lambda_1^2 & \lambda_2^2 \ldots & \lambda_r^2 \\ \vdots & \vdots & \vdots \\ \lambda_1^r & \lambda_2^r \ldots & \lambda_r^r \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \\ \vdots \\ m_r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$
. By properties of the determinant, we have $\det(\begin{pmatrix} \lambda_1 & \lambda_2 \ldots & \lambda_r \\ \lambda_1^2 & \lambda_2^2 \ldots & \lambda_r^2 \\ \vdots & \vdots & \vdots \\ \lambda_1^r & \lambda_2^r \ldots & \lambda_r^r \end{pmatrix}) = \lambda_1 \ldots \lambda_r \det(\begin{pmatrix} 1 & 1 \ldots & 1 \\ \lambda_1 & \lambda_2 \ldots & \lambda_r \\ \vdots & \vdots & \vdots \\ \lambda_1^{r-1} & \lambda_2^{r-1} \ldots & \lambda_r^{r-1} \end{pmatrix})$.
The preceding matrix is a Vandermonde matrix, and is invertible because all λ_i are distinct. Therefore, $m_i = 0$ for all i, which is a contradiction because $m_i \geq 1$ by definition. Therefore, all eigenvalues of A must be 0, so that $c_A(x) = x^n$. This says that that $m_A(x) = x^N$ for some N, i.e. A is nilpotent.

5. Let J be the Jordan canonical form of A, so that $A = SJS^{-1}$ for some matrix S. Our first observation is that matrix multiplication is continuous, because the entries of SJS^{-1} are just given by polynomials in the entries of S,J,and S^{-1}. Therefore, if we can find a sequence of diagonalizable matrices D_n such that $\lim_{n\to\infty} D_n = J$, we have by continuity that $A = \lim_{n\to\infty} SD_nS^{-1}$. Therefore, it suffices to prove this for a Jordan block, because if $J = \mathrm{Diag}(J_1, \ldots, J_r)$ and $\lim_{n\to\infty} D_{i,n} = J_i$, we have $\lim_{n\to\infty} \mathrm{Diag}(D_{1,n}, \ldots, D_{r,n}) = J$. If J is Jordan block of size n and eigenvalue λ, pick n distinct numbers $\varepsilon_1, \ldots, \varepsilon_n$ and consider the matrix D_n obtained by adding ε_i/n to the diagonal entries of J. Then D_n has distinct eigenvalues (because they're the diagonal entries!) and therefore D_n is diagonalizable. Clearly $\lim_{n\to\infty} D_n = J$, so we're done.